ON SCHAUDER ESTIMATES FOR A CLASS OF NONLOCAL FULLY NONLINEAR PARABOLIC EQUATIONS

HONGJIE DONG AND HONG ZHANG

Abstract. We obtain Schauder estimates for a class of concave fully nonlinear nonlocal parabolic equations of order $\sigma \in (0,2)$ with rough and non-symmetric kernels. As a application, we prove that the solution to a translation invariant equation with merely bounded data is C^σ in x variable and Λ^1 in t variable, where Λ^1 is the Zygmund space.

1. introduction

This paper is devoted to the study of Schauder estimates for a class of concave fully nonlinear nonlocal parabolic equations. There is a vast literature on Schauder estimates for classical elliptic and parabolic equations, for instance, see [13, 18, 4]. Since the work by Caffarelli and Silvestre [1, 3, 2], nonlocal equations, which naturally arise from models in physics, engineering, and finance that involve long range interactions (for instance, see [9]), attract an increasing level of interest recently. An example of nonlocal operators, which is associated with pure jump processes (see, for instance, [19]), is the following

$$
\begin{aligned}
L_a u &= \int_{\mathbb{R}^d} \big(u(t,x+y) - u(t,x) - y^T Du(t,x)\big) K_a(t,x,y)\, dy \quad \text{for } \sigma \in (1,2),\\
L_a u &= \int_{\mathbb{R}^d} \big(u(t,x+y) - u(t,x) - y^T Du(t,x)\chi_{B_1}\big) K_a(t,x,y)\, dy \quad \text{for } \sigma = 1\\
&\quad \text{with} \quad \int_{S_r} y K_a(t,x,y)\, ds = 0 \quad \forall\, r>0, \qquad (1.1)\\
L_a u &= \int_{\mathbb{R}^d} \big(u(t,x+y) - u(t,x)\big) K_a(t,x,y)\, dy \quad \text{for } \sigma \in (0,1),
\end{aligned}
$$

where

$$
K_a \in \mathcal{L}_0 := \Big\{ K : \frac{\lambda}{|y|^{d+\sigma}} \le K(t,x,y) \le \frac{\Lambda}{|y|^{d+\sigma}} \Big\}
$$

for some ellipticity constants $0 < \lambda \le \Lambda$, with no regularity assumption imposed with respect to the y variable. This type of nonlocal operator was first considered by Komatsu [16], Mikulevičius and Pragarauskas [19, 20], and later by Dong and Kim [11, 10], and Schwab and Silvestre [22], to name a few.

The fully nonlinear nonlocal parabolic equation that we are interested in is of the form

$$
u_t = \inf_{a \in \mathcal{A}} (L_a u + f_a) \quad \text{in } (-1,0) \times B_1, \qquad (1.2)
$$

where $K_a \in \mathcal{L}_0$ for $a \in \mathcal{A}$ and $\mathcal{A}$ is an index set. For fully nonlinear second-order equations with $f_a \equiv 0$, the celebrated $C^{2,\alpha}$ estimate was established independently

H. Dong and H. Zhang were partially supported by the NSF under agreement DMS-1056737.

by Evans [12] and Krylov [17] in early nineteen-eighties. Nonhomogeneous second-order equations were considered a bit later by Safonov [21]. Recently, Caffarelli and Silvestre [2] investigated the nonlocal version of Evans-Krylov theorem with translation invariant and symmetric kernels, i.e., $K_a(x,y) = K_a(y) = K_a(-y)$, satisfying additional regularity assumptions

$$[K_a]_{C^2(\mathbb{R}^d\setminus B_\rho)} \le \Lambda(2-\sigma)\rho^{-d-\sigma-2}. \tag{1.3}$$

More recently, their result was extended to nonhomogeneous fully nonlinear elliptic equations by Jin and Xiong [14] by using a recursive Evans-Krylov theorem. At almost the same time, Serra [23] removed the regularity assumption (1.3) and proved the Evans-Krylov theorem and Schauder estimates with symmetric kernels. His proof relies on a Liouville type theorem and a blow-up analysis. In this paper, we do not assume that the kernels are symmetric, which is certainly more general than the kernels considered in [2, 14, 23]. Specifically, when the kernels are symmetric, (1.1) is satisfied automatically, and

$$L_a u = \frac{1}{2}\int_{\mathbb{R}^d} \big(u(x+y) + u(x-y) - 2u(x)\big) K_a \, dy,$$

the right-hand side of which is the operator considered in [2, 14, 23].

For equations with non-symmetric kernels, Dong and Kim [10, 11] proved L_p and Schauder estimates for linear elliptic equations. Chang-Lara and Dávila [7, 8] considered nonlocal parabolic equations with non-symmetric kernels and critical drift, and proved the corresponding C^α and $C^{1,\alpha}$ estimate. Recently in [5], they proved a version of the Evan-Krylov theorem for concave nonlocal parabolic equations with critical drift, where they assumed the kernels to be non-symmetric but translation invariant and smooth (1.3). We also mention that Schauder estimates for linear nonlocal parabolic equations were studied in [15, 20].

The objective of this paper is twofold. First we extend the previous results in [23, 5, 14, 15] to include concave nonlocal parabolic equations with non-symmetric rough kernels. More specifically, for any small α, if f_a and $K_a(t,x,y)$ are C^α in x and $C^{\alpha/\sigma}$ in t, then we have the following $C^{1+\alpha/\sigma,\sigma+\alpha}$ a priori estimate of any smooth solution u to (1.2).

Theorem 1.1. *Let $\sigma \in (0,2)$, $0 < \lambda \le \Lambda < \infty$, and $\mathcal{A}$ be an index set. There is a constant $\hat\alpha \in (0,1)$ depending on $d, \sigma, \lambda,$ and Λ so that the following holds. Let $\alpha \in (0,\hat\alpha)$ such that $\sigma+\alpha$ is not an integer. Assume $K_a \in \mathcal{L}_0$ and satisfies (1.1) when $\sigma = 1$, and*

$$\big|K_a(t,x,y) - K_a(t',x',y)\big| \le A\big(|x-x'|^\alpha + |t-t'|^{\alpha/\sigma}\big)\frac{\Lambda}{|y|^{d+\sigma}}, \tag{1.4}$$

where $A \ge 0$ is a constant. Suppose $u \in C^{1+\alpha/\sigma,\sigma+\alpha}(Q_1) \cap C^{\alpha/\sigma,\alpha}((-1,0)\times\mathbb{R}^d)$ is a solution of

$$u_t = \inf_{a\in\mathcal{A}}(L_a u + f_a) \quad \text{in } Q_1, \tag{1.5}$$

where $f_a \in C^{\alpha/\sigma,\alpha}(Q_1)$ satisfying

$$C_0 := \sup_{a\in\mathcal{A}}[f_a]_{\alpha/\sigma,\alpha;Q_1} < \infty, \qquad \sup_{(t,x)\in Q_1}\Big|\inf_{a\in\mathcal{A}} f_a(t,x)\Big| < \infty.$$

Then,

$$[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{1/2}} \le C\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d} + CC_0, \tag{1.6}$$

where $C > 0$ is a constant depending only on $d, \lambda, \Lambda, \alpha, A,$ and σ.

Note that $\|\cdot\|_{\alpha/\sigma,\alpha;\Omega}$ is the Hölder norm of order α/σ in t and α in x with underlying domain Ω. We also used Q_r to denote the parabolic cylinder with radius r centered at the origin. For precise definitions, see Section 2. As pointed out in [23], the $C^{\alpha/\sigma,\alpha}$ Hölder norm of u on the right-hand side of (1.6) is necessary and cannot be replaced by the L_∞ norm or any lower-order Hölder norm of u.

Roughly speaking, the proof of Theorem 1.1 can be divided into three steps. First we prove a Liouville type theorem for solutions in $(-\infty,0)\times\mathbb{R}^d$. For the classical PDEs, we generally apply interpolation and iteration to obtain $C^{1,\alpha}$ and $C^{2,\alpha}$ estimates. The nature of nonlocal operator is quite different from the classical operator. One notable feature is that the boundary data is prescribed on the complement of the domain where the equation is satisfied, which makes it difficult to implement interpolation and iteration to deal with the nonlocal operator. However, if we assume that (1.2) is satisfied in $(-\infty,0)\times\mathbb{R}^d$, then we do not need to worry about boundary data any more, which is the advantage of considering an equation satisfied in the whole space. Second, we prove the a priori estimate for equations with translation invariant kernels by combining the Liouville theorem and a blow-up analysis. Particularly in this step, the extension from symmetric kernels to non-symmetric kernels is non-trivial. A key idea in the classical Evans-Krylov theorem for $F(D^2u)=0$ is that, since the function F is concave, any second directional derivative $D^2_{ee}u$ is a subsolution. It is relatively easy to adapt this idea to the nonlocal equation with symmetric kernels, because the centered second-order difference appears in the definition of the operator. For nonsymmetric kernels, some new ideas are required to obtain a similar subsolution as in the symmetric case. Moreover, the dependence of the t variable also makes the proof more involved. Finally, we implement a more or less standard perturbation argument to treat the general case.

The second objective of this paper is to consider the end point situation when $\alpha=0$. For second-order elliptic equations, even the Poisson equation $\Delta u=f$, when f is merely bounded, it is well known that u may fail to be $C^{1,1}$. However, this is not the case for nonlocal equations. In the case when $\sigma\neq 1$ and the kernels are independent of t and x, we prove a priori C^σ estimate in the x variable and Λ^1 estimate in the t variable when f_a is merely bounded. Here Λ^1 is the Zygmund space. To our best knowledge, such result is new even for nonlocal elliptic equations with symmetric kernels.

Theorem 1.2. *Let $\sigma\neq 1$. Assume that u is C^σ in x, Λ^1 in t and satisfies* (1.2) *in $(-\infty,0)\times\mathbb{R}^d$ with K_a independent of t and x. When $\sigma\in(1,2)$, we also assume that Du is $C^{(\sigma-1)/\sigma}$ in t. Then there exists a constant C depending on d,λ,Λ, and σ such that for $\sigma>1$,*

$$[u]^t_{\Lambda^1}+[u]^*_\sigma+[Du]^t_{\frac{\sigma-1}{\sigma}}\le C\sup_a\|f_a\|_{L_\infty};$$

for $\sigma<1$,

$$[u]^t_{\Lambda^1}+[u]^*_\sigma\le C\sup_a\|f_a\|_{L_\infty},$$

where all the norms are taken in $\mathbb{R}^{d+1}_0:=(-\infty,0)\times\mathbb{R}^d$.

Here $[\,\cdot\,]^*$, $[\,\cdot\,]^t_\alpha$, and $[\,\cdot\,]^t_{\Lambda^1}$ are the Hölder semi-norms in x, t, and Zygmund semi-norm in t, respectively. See the precise definitions in Section 2.

The proof of Theorem 1.2 is based on a perturbation type argument using Campanato's approach. We first refine the estimate in Theorem 1.1 when the operator is translation invariant. In particular, we replace $\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d}$ on the right-hand side of (1.6) by

$$\|u\|_{\alpha/\sigma,\alpha;Q_1}+\sum_{j=1}^{\infty}2^{-j\sigma}[u]_{\alpha/\sigma,\alpha;(-1,0)\times(B_{2^j}\setminus B_{2^{j-1}})}. \tag{1.7}$$

The advantage of the replacement will be explained below. Another important ingredient in the proof is the fact that for $\sigma>1$

$$[u]^t_{\Lambda^1(\mathbb{R}^{d+1}_0)}+[u]^*_{\sigma;\mathbb{R}^{d+1}_0}+[Du]^t_{\frac{\sigma-1}{\sigma};\mathbb{R}^{d+1}_0}\le C\sup_{r>0}\sup_{(t,x)\in\mathbb{R}^{d+1}_0}E[u;Q_r(t,x)], \tag{1.8}$$

and for $\sigma<1$,

$$[u]^t_{\Lambda^1(\mathbb{R}^{d+1}_0)}+[u]^*_{\sigma;\mathbb{R}^{d+1}_0}\le C\sup_{r>0}\sup_{(t,x)\in\mathbb{R}^{d+1}_0}E[u;Q_r(t,x)], \tag{1.9}$$

where

$$E[u;Q_r(t,x)]:=\inf_{p\in\mathcal{P}}r^{-\sigma}\|u-p\|_{L_\infty(Q_r(t,x))},$$

$\mathcal{P}$ is the set of polynomial of degree $[\sigma]$ (i.e., the integer part of σ) in x and linear in t, and $Q_r(t,x)$ is the parabolic cylinder with center (t,x); see (2.1). Therefore, instead of directly estimating

$$[u]^t_{\Lambda^1}+[u]^*_\sigma+[Du]^t_{\frac{\sigma-1}{\sigma}}\quad(\text{or }[u]^t_{\Lambda^1}+[u]^*_\sigma),$$

we estimate $E[u;Q_r(t,x)]$ for any fixed r and (t,x). More specifically, without loss of generality, we set $(t,x)=(0,0)$ and let v_K solve the homogeneous equation

$$\begin{cases}\partial_t v_K=\inf_{a\in\mathcal{A}}L_a v_K & \text{in } Q_{2R}\\ v_K=g_K:=\max\{-K,\min\{u-p,K\}\} & \text{in } (-(2R)^\sigma,0)\times B^c_{2R}\end{cases},$$

where K is a large constant, p is a carefully chosen linear function, and $R>2r$ is a constant to be determined. Now we apply Theorem 1.1 to v_K and control $[v_K]_{1+\alpha/\sigma,\alpha+\sigma;Q_{R/2}}$ by using scaling argument and replacing $\|v_K\|_{\alpha/\sigma,\alpha}$ by (1.7). It is easily seen that in each cylindrical domain $(-R^\sigma,0)\times(B_{2^jR}\setminus B_{2^{j-1}R})$, the Hölder norm of g_K is bounded and independent of K, but globally it depends on K and goes to infinity as $K\to\infty$. This is also the advantage of decomposing the domain into annuli. We then set q_K to be the first-order Taylor expansion of v_K and we estimate

$$\|u-p-q_K\|_{L_\infty(Q_r)}\le\|u-p-v_K\|_{L_\infty(Q_r)}+\|v_K-q_K\|_{L_\infty(Q_r)},$$

where the first term is bounded by CR^σ due to the Aleksandrov-Bakelman-Pucci estimate and second term is controlled by $[v_K]_{1+\alpha/\sigma,\alpha+\sigma;Q_r}$. Finally, we are able to obtain

$$r^{-\sigma}\|u-p-q_K\|_{L_\infty(Q_r)}\le C(r/R)^\alpha\big([u]^*_\sigma+[u]^t_{\Lambda^1}+[Du]^t_{\frac{\sigma-1}{\sigma}}\big)+C(R/r)^\sigma\|f\|_{L_\infty}.$$

By setting $R=Mr$, using (1.8) (or (1.9)), and taking M sufficiently large, the terms involving u on the right-hand side above are absorbed in the left-hand side.

It seems that new ideas are needed to deal with the case when $\sigma=1$, because in this case we expect that $u\in\Lambda^1$ in x and it is unclear to us how to choose p in order to get a good estimate of $u-p$ in $(-R^\sigma,0)\times(B_{2^jR}\setminus B_{2^{j-1}R})$.

We localize Theorem 1.2 to obtain the following corollary.

Corollary 1.3. *Let $\sigma \neq 1$. Assume that u is $C^\sigma(Q_1)$ in x, $\Lambda^1(Q_1)$ in t, and satisfies*

$$u_t = \inf_{a\in\mathcal{A}} (L_a u + f_a) \quad \text{in } Q_1.$$

When $\sigma \in (1,2)$, we also assume that Du is $C^{(\sigma-1)/\sigma}(Q_1)$ in t. Then for $\sigma > 1$,

$$[u]^*_{\sigma;Q_{1/2}} + [u]^t_{\Lambda^1(Q_{1/2})} + [Du]_{\frac{\sigma-1}{\sigma};Q_{1/2}} \le C\Big(\sup_{a\in\mathcal{A}} \|f_a\|_{L_\infty(Q_1)} + \|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}\Big);$$

and for $\sigma < 1$,

$$[u]^*_{\sigma;Q_{1/2}} + [u]^t_{\Lambda^1(Q_{1/2})} \le C\Big(\sup_{a\in\mathcal{A}} \|f_a\|_{L_\infty(Q_1)} + \|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)}\Big).$$

We remark that by viewing solutions to elliptic equations as steady state solutions to parabolic equations, from Theorems 1.1, 1.2, and Corollary 1.3, we obtain the corresponding results for nonlocal elliptic equations with nonsymmetric and rough kernels.

The organization of this paper is as follows. In the next section, we introduce some notation and preliminary results that are necessary in the proof of our main results. We prove the Liouville theorem in Section 3 and Theorem 1.1 in Section 4. In Section 5, we apply Theorem 1.1 to prove Theorem 1.2.

2. Notation and preliminary results

In this section, we introduce some notation which will be used throughout this paper and some preliminary results which are useful in our proof. We use $B_r(x)$ to denote the Euclidean ball in $\mathbb{R}^d$ with center x and radius r. The parabolic cylinder $Q_r(t,x)$ is defined as follows

$$Q_r(t,x) = (t - r^\sigma, t) \times B_r(x). \tag{2.1}$$

We simply use Q_r to denote $Q_r(0,0)$ and $\mathbb{R}^{d+1}_0 := (-\infty, 0)\times\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^{d+1}$ and we define the Hölder semi-norm as follows: for any $\alpha, \beta \in (0,1]$, and function f,

$$[f]_{\alpha,\beta;\Omega} = \sup\Big\{\frac{|f(t,x) - f(s,y)|}{\max(|x-y|^\alpha, |t-s|^\beta)} : (t,x),(s,y)\in\Omega, (t,x)\neq(s,y)\Big\}.$$

We denote

$$\|f\|_{\alpha,\beta;\Omega} = \|f\|_{L_\infty(\Omega)} + [f]_{\alpha,\beta;\Omega}.$$

For any nonnegative integers m and n,

$$\|f\|_{m+\alpha,n+\beta;\Omega} = \|f\|_{L_\infty(\Omega)} + [D^m f]_{\alpha,\beta;\Omega} + [\partial_t^n f]_{\alpha,\beta;\Omega}.$$

The spaces corresponding to $\|\cdot\|_{\alpha,\beta;\Omega}$ and $\|\cdot\|_{m+\alpha,n+\beta;\Omega}$ are denoted by $C^{\alpha,\beta}(\Omega)$ and $C^{m+\alpha,n+\beta}(\Omega)$, respectively. Next, for any $\alpha, \beta \in (0,1]$, we define the Hölder semi-norms only with respect to x or t

$$[f]^*_{\alpha;\Omega} = \sup\Big\{\frac{|f(t,x) - f(t,y)|}{|x-y|^\alpha} : (t,x),(t,y)\in\Omega, x\neq y\Big\},$$

$$[f]^t_{\beta;\Omega} = \sup\Big\{\frac{|f(t,x) - f(s,x)|}{|t-s|^\beta} : (t,x),(s,x)\in\Omega, t\neq s\Big\}.$$

When $\sigma = k + \alpha$ with some integer $k \ge 1$,

$$[f]^*_{\sigma;\Omega} = [D^k f]^*_{\alpha;\Omega}.$$

For $\alpha \in (0,2)$, we define the Lipschitz-Zygmund norm

$$\|u\|_{\Lambda^\alpha} := \|u\|_{L_\infty} + \sup_{|h|>0} |h|^{-\alpha} \|u(\cdot + h) + u(\cdot - h) - 2u(\cdot)\|_{L_\infty}.$$

We say $u \in \Lambda^\alpha$ if $\|u\|_{\Lambda^\alpha} < \infty$.

For simplicity of notation, we denote

$$\delta u(t,x,y) = \begin{cases} u(t,x+y) - u(t,x) - y^T Du(t,x) & \text{for } \sigma \in (1,2), \\ u(t,x+y) - u(t,x) - y^T Du(t,x)\chi_{B_1} & \text{for } \sigma = 1, \\ u(t,x+y) - u(t,x) & \text{for } \sigma \in (0,1). \end{cases}$$

The Pucci extremal operator is defined as follows: for $\sigma \neq 1$

$$\mathcal{M}^+ u(t,x) = \int_{\mathbb{R}^d} \left(\Lambda \delta u(t,x,y)^+ - \lambda \delta u(t,x,y)^-\right) \frac{2-\sigma}{|y|^{d+\sigma}}\, dy,$$
$$\mathcal{M}^- u(t,x) = \int_{\mathbb{R}^d} \left(\lambda \delta u(t,x,y)^+ - \Lambda \delta u(t,x,y)^-\right) \frac{2-\sigma}{|y|^{d+\sigma}}\, dy.$$

When $\sigma = 1$, the extremal operator cannot be written out explicitly, due to the condition (1.1). Nevertheless, we do not use exact representation directly and define the extremal operator by

$$\mathcal{M}^+ u = \sup_a L_a u \qquad \text{and} \quad \mathcal{M}^- u = \inf_a L_a u,$$

where the infimum (or supremum) is taken with respect to all L_a's with kernels K_a satisfying (1.1).

We recall the weak Harnack inequality of [22, Theorem 6.1].

Proposition 2.1. *Assume that $0 < \sigma_0 \le \sigma < 2$ and $C > 0$ is a constant. Let u be a function such that*

$$u_t - \mathcal{M}^- u \ge -C \quad \text{in} \quad Q_1, \quad u \ge 0 \quad \text{in} \quad (-1,0) \times \mathbb{R}^d.$$

Then there are constants $C_1 > 0$ and $\varepsilon_1 \in (0,1)$ depending only on $\sigma_0, \lambda, \Lambda$, and d, such that

$$\Big(\int_{(-1,-2^{-\sigma}) \times B_{1/4}} u^{\varepsilon_1}\, dx\, dt \Big)^{1/\varepsilon_1} \le C_1 \Big(\inf_{Q_{1/4}} u + C \Big).$$

From Proposition 2.1, we obtain the following corollary for $\sigma \in (1,2)$, the proof of which is provided in the appendix.

Corollary 2.2. *Let $\sigma \in (1,2)$, $C > 0$ be a constant, and u satisfy*

$$u_t - \mathcal{M}^- u \ge -C \quad \text{in} \quad Q_{2r}, \quad u \ge 0 \quad \text{in} \quad (-(2r)^\sigma, 0) \times \mathbb{R}^d.$$

Let ε_1 be the constant in Proposition 2.1. For any $r, \delta \in (0,1)$, denote $\tilde{Q}_{\delta r} = (-r^\sigma, -(\delta r)^\sigma) \times B_r$. Then we have

$$r^{-(d+\sigma)/\varepsilon_1} \Big(\int_{\tilde{Q}_{\delta r}} u^{\varepsilon_1}\, dx\, dt \Big)^{1/\varepsilon_1} \le C_2 \Big(\inf_{Q_{\delta r/2}} u + C r^\sigma \Big),$$

where $C_2 > 0$ is a constant depending only on $\delta, \sigma_0, \lambda, \Lambda$, and d.

We state the following local boundedness estimate from [6, Corollary 6.2].

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^d$, $t_1 < t_2$, and u satisfy*

$$u_t - \mathcal{M}^+ u \le 0 \quad \text{in} \quad (t_1, t_2] \times \Omega.$$

Then for any $(t_1', t_2] \times \Omega' \subset\subset (t_1, t_2] \times \Omega$,

$$\sup_{\Omega' \times (t_1', t_2]} u^+ \le C \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \frac{u^+}{1 + |x|^{d+\sigma}} \, dx \, dt,$$

where C depends on $\Omega, \Omega', t_1, t_2$, and t_1'.

Let us point out that the kernels considered in [6] are more general than our kernels. Specifically, Chang-Lara and Dávila considered when $\sigma \in [1, 2)$

$$Lu = (2 - \sigma) \int_{\mathbb{R}^d} \hat{\delta} u(x, y) K(y) \, dy + b \cdot Du(x),$$

where $\hat{\delta} u(x, y) = u(x + y) - u(x) - Du(x) y \chi_{B_1}$, $K(y) \in \mathcal{L}_0$, and for some $\beta > 0$

$$\sup_{r \in (0,1)} r^{\sigma - 1} \left| b + (2 - \sigma) \int_{B_1 \setminus B_r} y K(y) \, dy \right| \le \beta. \tag{2.2}$$

Note that for $\sigma > 1$, since

$$\delta u(x, y) = \hat{\delta} u(x, y) - Du(x) y \chi_{B_1^c},$$

we can rewrite our operator and get

$$b = -(2 - \sigma) \int_{B_1^c} y K(y) \, dy.$$

Obviously, $|b| \le C$, where C depends d, σ, and Λ, and it is easy to check that (2.2) holds for b and K above.

The next proposition is [22, Theorem 7.1].

Proposition 2.4. *Let u satisfy in Q_1*

$$u_t - \mathcal{M}^+ u \le C_0 \quad \text{and} \quad u_t - \mathcal{M}^- u \ge -C_0.$$

Then there are constants $\gamma \in (0, 1)$ and $C_1 > 0$ only depending on d, σ, λ, and Λ such that

$$[u]_{\gamma/\sigma, \gamma; Q_{1/2}} \le C_1 \|u\|_{L_\infty((-1,0); L_1(\omega_\sigma))} + C C_0.$$

Here

$$\|u\|_{L_\infty((-1,0); L_1(\omega_\sigma))} = \sup_{t \in (0,1)} \int_{\mathbb{R}^d} \frac{|u(t, x)|}{1 + |x|^{d+\sigma}} \, dx.$$

Note that we replaced $\|u\|_{L_\infty((-1,0) \times \mathbb{R}^d)}$ by $\|u\|_{L_\infty((-1,0); L_1(\omega_\sigma))}$, which follows from a simple localization argument. See, for instant, [6, Corollary 7.1]. In the sequel, we always assume $\gamma < \sigma$.

We finish this section by proving the following global Hölder estimate.

Lemma 2.5. *Let u satisfy in Q_1*

$$u_t - \mathcal{M}^+ u \le C_0 \quad \text{and} \quad u_t - \mathcal{M}^- u \ge -C_0,$$

where C_0 is a constant and $u \equiv 0$ in $\mathbb{R}^{d+1}_0 \setminus Q_1$. Then there exists a constant $\alpha \in (0, 1)$ depending on d, λ, Λ, and σ, so that

$$[u]_{\alpha/\sigma, \alpha; Q_1} \le C C_0,$$

where C depends on d, λ, Λ, and σ.

Proof. Thanks to the interior Hölder estimate Proposition 2.4, it suffices to prove the estimate near the parabolic boundary of Q_1. We consider the lateral boundary and bottom separately. Define $\phi : \mathbb{R}^d \to \mathbb{R}^+$ as

$$\phi(x) = \begin{cases} x_d^\beta & \text{for } x_d > 0 \\ 0 & \text{for } x_d \le 0 \end{cases},$$

where $\beta \in (0,1)$. We claim that for sufficiently small $\beta \in (0,\sigma)$ depending on d, λ, Λ, and σ, we have

$$\mathcal{M}^+\phi(x) < -\hat{C}x_d^{\beta-\sigma} \quad \text{in} \quad \{x_d > 0\},$$

where $\hat{C}$ depends on d, λ, Λ, and σ. By scaling, it is obvious that

$$\mathcal{M}^+\phi(x) = x_d^{\beta-\sigma}\mathcal{M}^+\phi(e),$$

where $e = (0,0,\ldots,0,1)$. Therefore, we only need to estimate $\mathcal{M}^+\phi(e)$.

Case 1: $\sigma > 1$. By definition,

$$\begin{aligned}
&\mathcal{M}^+\phi(e)\\
&= (2-\sigma)x_d^{\beta-\sigma}\left(\int_{\{y_d>-1\}} + \int_{\{y_d<-1\}}\right)\big(\Lambda(\delta\phi(e,y))^+ - \lambda(\delta\phi(e,y))^-\big)\frac{1}{|y|^{d+\sigma}}\,dy\\
&=: (2-\sigma)x_d^{\beta-\sigma}(\mathrm{I}_1 + \mathrm{I}_2).
\end{aligned}$$

When $y_d > -1$, by concavity, it follows that

$$\phi(e+y) = (1+y_d)^\beta < 1 + \beta y_d \quad \text{and} \quad \delta\phi(e,y) \le 0.$$

Therefore,

$$\begin{aligned}
\mathrm{I}_1 &\le \int_{\{|y_d|<1\}} \lambda\delta\phi(e,y)\frac{1}{|y|^{d+\sigma}}\,dy\\
&= \lambda\int_{y_d\in(0,1)} \big((1+y_d)^\beta + (1-y_d)^\beta - 2\big)\frac{1}{|y|^{d+\sigma}}\,dy.
\end{aligned}$$

Notice that for any $s \in (-1,1)$,

$$(1+s)^\beta < 1 + \beta s + \frac{\beta(\beta-1)}{2}s^2 + \frac{\beta(\beta-1)(\beta-2)}{6}s^3,$$

which implies that for $y_d \in (0,1)$

$$(1+y_d)^\beta + (1-y_d)^\beta - 2 < \beta(\beta-1)y_d^2.$$

Therefore,

$$\mathrm{I}_1 < \lambda\beta(\beta-1)\int_{y_d\in(0,1)}\frac{y_d^2}{|y|^{d+\sigma}}\,dy = C_1\beta(\beta-1),$$

where C_1 depends on d, λ, and σ.

Now we turn to I_2. Since $\phi(e+y) = 0$ when $y_d < -1$, we have

$$\begin{aligned}
\mathrm{I}_2 &= \int_{\{y_d<-1\}} \frac{\Lambda(-\beta y_d - 1)^+ - \lambda(-\beta y_d - 1)^-}{|y|^{d+\sigma}}\,dy\\
&\le \int_{\{y_d<-1/\beta\}} \frac{\Lambda(-\beta y_d - 1)}{|y|^{d+\sigma}}\,dy = C_2\beta^\sigma,
\end{aligned}$$

where C_2 depends on Λ, d, and σ. Thanks to the estimates of I_1 and I_2 above, it follows that

$$\mathcal{M}^+\phi(e) \le C_1\beta(\beta-1) + C_2\beta^\sigma.$$

By choosing β sufficiently small depending on λ, Λ, d, and σ so that

$$C_1(\beta-1) + C_2\beta^{\sigma-1} \le -C_1/2,$$

the claim is proved.

Case 2: $\sigma < 1$. Let I_1 and I_2 be defined as before. Since $\phi(x) = 0$ for $x_d < 0$, we get

$$\mathrm{I}_2 = -(2-\sigma)\int_{\{y_d<-1\}} \frac{\lambda}{|y|^{d+\sigma}}\,dy = -C_3, \tag{2.3}$$

where $C_3 > 0$ depends on σ, λ, and d. For I_1, we have

$$\begin{aligned}\mathrm{I}_1 &= (2-\sigma)\int_{\{y_d>0\}} \frac{\Lambda((1+y_d)^\beta - 1)}{|y|^{d+\sigma}}\,dy - (2-\sigma)\int_{y_d\in(-1,0)} \frac{\lambda(1-(1+y_d)^\beta)}{|y|^{d+\sigma}}\,dy\\ &\le (2-\sigma)\Lambda\int_{\{y_d>0\}} \frac{(1+y_d)^\beta - 1}{|y|^{d+\sigma}}\,dy \to 0 \quad \text{as} \quad \beta\to 0\end{aligned}$$

by the monotone convergence theorem. Therefore, we can choose β small depending on Λ, λ, d, and σ so that

$$\mathcal{M}^+\phi(e) \le -C_3/2.$$

The claim is proved.

Case 3: $\sigma = 1$. In this case, we still have (2.3). For I_1, we notice that integrand in the region $\{-1 < y_d < 0\} \cup \{|y| < 1\}$ is negative, and

$$\int_{\{y_d>0\}\cap\{|y|>1\}} \frac{(1+y_d)^\beta - 1}{|y|^{d+\sigma}}\,dy \to 0 \quad \text{as} \quad \beta \to 0$$

by the monotone convergence theorem. Thus the claim follows as well.

Now we are ready to consider u near the lateral boundary. By a translation and rotation of the coordinates, we replace the ball B_1 by $B_1(e)$ and estimate u near the origin. Define the barrier function $\psi(t,x) = \frac{C_0}{2^{\beta-\sigma}\hat{C}}\phi(x)$. Obviously,

$$\partial_t\psi(t,x) - \mathcal{M}^+\psi(t,x) = -\frac{C_0}{2^{\beta-\sigma}\hat{C}}\mathcal{M}^+\phi(x) > \frac{C_0}{2^{\beta-\sigma}}x_d^{\beta-\sigma} > C_0$$

when $x \in B_1(e)$. On the other hand, $\psi \ge 0$ in $\mathbb{R}\times\mathbb{R}^d$. Since

$$u_t - \mathcal{M}^+u \le C_0$$

and $u \equiv 0$ outside Q_1, by the comparison principle,

$$u(t,x) \le \psi(t,x) = \frac{C_0}{2^{\beta-\sigma}\hat{C}}x_d^\beta.$$

By considering $-u$ instead of u, we have $u \ge -\frac{C_0}{2^{\beta-\sigma}\hat{C}}x_d^\beta$. Hence, around the origin $|u| \le C|x|^\beta$. By rotation of the coordinate, we obtain the estimate near the lateral boundary.

For the bottom, let $\tilde{\phi} = C_0(t_0+1)$ so that $\tilde{\phi}(-1) = 0$ and $\tilde{\phi}'(t) = C_0$. This yields that

$$\partial_t\tilde{\phi} - \mathcal{M}^+\phi = C_0.$$

Moreover, $\tilde{\phi} \geq 0$ in $(-1,0) \times \mathbb{R}^d$. By the comparison principle again, $u \leq \phi$ in Q_1. In particular, near the bottom $u \leq C_0(t+1)$, which further implies $|u| \leq C_0(t+1)$ by symmetry.

Combining the estimates of lateral boundary and bottom with the interior Hölder estimate, we prove the lemma. $\square$

3. A Liouville theorem

The aim of this section is to prove the following Liouville theorem for the fully nonlinear parabolic nonlocal equation with non-symmetric kernels. The elliptic version for symmetric kernels was established in [23].

Theorem 3.1. *Let $\sigma_0 \in (0,1)$ and $\sigma \in [\sigma_0, 2)$. There is a constant $\hat{\alpha} \in (0,1/2)$ depending on d, λ, Λ, and σ_0 such that the following statement holds. Let $\alpha \in (0, \hat{\alpha})$ be such that $[\sigma + \hat{\alpha}] < \sigma + \alpha$ and if $u \in C_{loc}^{1+\frac{\alpha}{\sigma}, \sigma+\alpha}(\mathbb{R}_0^{d+1})$ satisfying the following properties:*

(i) For any $\beta \in [0, \sigma+\alpha]$ and $R \geq 1$, we have

$$[u]_{\beta/\sigma, \beta; Q_R} \leq N_0 R^{\sigma+\alpha-\beta}; \tag{3.1}$$

(ii) For any $(s,h) \in \mathbb{R}_0^{d+1}$, we have

$$\partial_t\big(u(\cdot+s, \cdot+h) - u\big) - \mathcal{M}^-\big(u(\cdot+s,\cdot+h) - u\big) \geq 0, \tag{3.2}$$

$$\partial_t\big(u(\cdot+s, \cdot+h) - u\big) - \mathcal{M}^+\big(u(\cdot+s,\cdot+h) - u\big) \leq 0; \tag{3.3}$$

For $\sigma > 1$, we further impose (iii) For any nonnegative measure μ in $\mathbb{R}^d$ with compact support, $\int_{\mathbb{R}^d} \delta u(t,x,h)\, d\mu(h)$ is a subsolution.

Then u is a polynomial of degree ν in x and 1 in t, where ν is the integer part of $\sigma + \alpha$.

Remark 3.2. As in [23], it is possible to relax Condition (i) in Theorem 3.1 by assuming that (3.1) is satisfied for any $\beta \in [0, \sigma+\alpha']$ and $R \geq 1$, where $\alpha' \in (0,\alpha)$ is a constant satisfying $\sigma + \alpha' > \nu$. A simple computation reveals that in the case we also require that $\sigma > 1 + \alpha - \alpha'$ when $\sigma > 1$ and $\nu = 1$; and $\sigma > \alpha - \alpha'$ when $\sigma < 1$ and $\nu = 0$.

To prove Theorem 3.1, we first present a few lemmas. Define

$$P(t,x) = \int_{\mathbb{R}^d} \big(\delta u(t,x,y) - \delta u(0,0,y)\big)^+ \frac{2-\sigma}{|y|^{d+\sigma}}\, dy,$$

$$N(t,x) = \int_{\mathbb{R}^d} \big(\delta u(t,x,y) - \delta u(0,0,y)\big)^- \frac{2-\sigma}{|y|^{d+\sigma}}\, dy.$$

Lemma 3.3. *Let $\hat{\alpha} \in (0,1/2)$ be a constant satisfying $\hat{\alpha} < \sigma/2$. Under the conditions (i) and (ii) of Theorem 3.1, for any $\kappa \geq 2$ and $l \in \mathbb{N} \cup \{0\}$, we have*

$$\sup_{Q_{\kappa^l}} \big(P + N + |u_t - u_t(0,0)|\big) \leq C N_0 \kappa^{\alpha l} \leq C N_0 \kappa^{\hat{\alpha} l} \tag{3.4}$$

and

$$[u_t]_{\gamma/\sigma, \gamma; Q_{1/2}} \leq C \kappa^{\hat{\alpha}}, \tag{3.5}$$

where C depends on d, λ, Λ, and σ.

Proof. We first estimate P and N assuming that $\nu = 2$. Fix $(t,x),(t',x') \in Q_1$ and set $l = |x-x'| + |t-t'|^\sigma$. By Condition (i), when $|y| < l$,

$$
\begin{aligned}
&|\delta u(t,x,y) - \delta u(t',x',y)| \\
&= \Big| \int_0^1 y \Big[Du(t,x+sy) - Du(t,x) - \big(Du(t',x'+sy) - Du(t',x')\big)\Big]\, ds \Big| \\
&\le C|y|^2 l^{\sigma+\alpha-2}[u]_{1+\alpha/\sigma,\sigma+\alpha;Q_2} \le CN_0|y|^2 l^{\sigma+\alpha-2}. \qquad (3.6)
\end{aligned}
$$

Similarly, when $|y| \ge l$,

$$
|\delta u(t,x,y) - \delta u(t',x',y)| \le C|y|^{\sigma+\alpha-1} l [u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{1+|y|}} \le CN_0 l|y|^{\sigma+\alpha-1}. \quad (3.7)
$$

Combining (3.6) and (3.7), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |\delta u(t,x,y) - \delta u(t',x',y)| \frac{2-\sigma}{|y|^{d+\sigma}}\, dy \\
&\le CN_0 l^{\sigma+\alpha-2} \int_{B_l} \frac{(2-\sigma)|y|^2}{|y|^{d+\sigma}}\, dy + CN_0 l \int_{\mathbb{R}^d \setminus B_l} \frac{(2-\sigma)|y|^{\sigma+\alpha-1}}{|y|^{d+\sigma}}\, dy \\
&\le CN_0 l^\alpha. \qquad (3.8)
\end{aligned}
$$

Hence $P, N \in C^{\alpha,\alpha/\sigma}(Q_1)$. Because $P(0,0) = N(0,0) = 0$, we have

$$
P(t,x) + N(t,x) \le CN_0 \quad \text{in } Q_1.
$$

By modifying the estimate above, we can prove the same estimate for P when $\nu = 0$ or 1.

We then use a scaling argument. Define $\hat{u}(t,x) = \eta^{-\alpha-\sigma} u(\eta^\sigma t, \eta x)$ for any $\eta > 1$. It is easily seen that $\hat{u}$ satisfies all the conditions in this lemma. Hence, we know that

$$
\int_{\mathbb{R}^d} \frac{(2-\sigma)\big|\delta\hat{u}(t,x,y) - \delta\hat{u}(0,0,y)\big|}{|y|^{d+\sigma}}\, dy \le CN_0 \quad \text{in } Q_1.
$$

Therefore,

$$
P(\eta^\sigma t, \eta x) + N(\eta^\sigma t, \eta x) \le CN_0 \eta^\alpha \quad \text{in } Q_1,
$$

which together with (3.2) and (3.3) implies (3.4).

To prove (3.5), we take $h = 0$ in (3.2) and (3.3), and then multiply them by $1/s$. By letting $s \to 0$, we know that u_t as well as $u_t - u_t(0,0)$ are sub and super-solutions at the same time. By Proposition 2.4, we obtain that $u_t \in C^{\gamma,\gamma/\sigma}(Q_{1/2})$ for some $\gamma > 0$ depending on d, λ, Λ, and σ, and

$$
[u_t]_{\gamma/\sigma,\gamma;Q_{1/2}} = [u_t - u_t(0,0)]_{\gamma/\sigma,\gamma;Q_{1/2}} \le C \sup_{t\in(-1,0)} \int_{\mathbb{R}^d} \frac{|u_t - u_t(0,0)|}{1+|x|^{\sigma+d}}\, dx.
$$

Using (3.4), for any $t \in (-1,0)$,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \frac{|u_t - u_t(0,0)|}{1+|x|^{d+\sigma}}\, dx\, dt \\
&\le \int_{B_1} \frac{|u_t - u_t(0,0)|}{1+|x|^{d+\sigma}}\, dx\, dt + \sum_{i=0}^\infty \int_{B_{\kappa^{i+1}} \setminus B_{\kappa^i}} \frac{|u_t - u_t(0,0)|}{1+|x|^{d+\sigma}}\, dx\, dt
\end{aligned}
$$

$$\le CN_0 \int_{B_1} \frac{1}{1+|x|^{d+\sigma}}\,dx\,dt + CN_0 \sum_{i=0}^{\infty} \int_{B_{\kappa^{i+1}}\setminus B_{\kappa^i}} \frac{\kappa^{\hat\alpha(i+1)}}{1+|x|^{d+\sigma}}\,dx\,dt$$
$$\le CN_0 + CN_0 \sum_{i=0}^{\infty} \kappa^{\hat\alpha(i+1)} \int_{\kappa^i}^{\kappa^{i+1}} \frac{r^{d-1}}{1+r^{d+\sigma}}\,dr$$
$$\le CN_0 + CN_0 \sum_{i=0}^{\infty} \frac{\kappa^{\hat\alpha}(1-\kappa^{-\sigma})}{1-\kappa^{\hat\alpha-\sigma}} \le CN_0\kappa^{\hat\alpha},$$

where C depends only on d,λ,Λ, and σ. Here we used the fact $\hat\alpha<\sigma/2$ and $\kappa\ge 2$ in the last inequality. Therefore, the lemma is proved. □

By dividing u by a CN_0, where C is the constant in (3.4), using Lemma 3.3 we have that for any $\kappa\ge 2$ and $l\in\mathbb{N}\cup\{0\}$,

$$\sup_{Q_{\kappa^l}} P \le \kappa^{\hat\alpha l}, \quad \sup_{Q_{\kappa^l}} N \le \kappa^{\hat\alpha l}. \tag{3.9}$$

We are going to prove inductively that there exists a sufficiently large $\kappa\ge 2$ and sufficiently small $\hat\alpha\in(0,1/2)$ such that

$$\sup_{Q_{\kappa^{-l}}} P \le \kappa^{-\hat\alpha l}, \quad \sup_{Q_{\kappa^{-l}}} N \le \kappa^{-\hat\alpha l} \quad \text{for any } l\in\mathbb{N}.$$

For a fixed $r\in(0,1)$, assume that P attains its maximum in $\overline{Q_r}$ at (t_0,x_0). Denote

$$\mathbb{A} := \big\{y : \delta u(t_0,x_0,y) - \delta u(0,0,y) > 0\big\}.$$

Then

$$P(t_0,x_0) = \int_{\mathbb{A}} \big(\delta u(t_0,x_0,y) - \delta u(0,0,y)\big)\frac{2-\sigma}{|y|^{d+\sigma}}\,dy,$$
$$N(t_0,x_0) = \int_{\mathbb{R}^d\setminus\mathbb{A}} \big(\delta u(t_0,x_0,y) - \delta u(0,0,y)\big)\frac{2-\sigma}{|y|^{d+\sigma}}\,dy.$$

We define

$$v(t,x) = \int_{\mathbb{A}} \big(\delta u(t,x,y) - \delta u(0,0,y)\big)\frac{2-\sigma}{|y|^{d+\sigma}}\,dy.$$

Notice that $v\le P$, and in particular $v\le 1$ in Q_1. Moreover, $P(t_0,x_0)=v(t_0,x_0)$. We denote $\mathfrak{v}=(1-v)^+$.

Lemma 3.4. *Suppose that $\hat\alpha\in(0,1/2)$ satisfying $\hat\alpha<\sigma/2$ and $\kappa\ge 2$. Then we have*

$$\mathfrak{v}_t - \mathcal{M}^-\mathfrak{v} \ge -C(\kappa^{\hat\alpha}-1) \quad \text{in } Q_{3/4}, \tag{3.10}$$

where C is a positive constant depending only on d,λ,Λ, and σ.

Proof. Since $v\le 1$ in Q_1, for any $(t,x)\in Q_1$ we have $\mathfrak{v}(t,x)=1-v(t,x)$, thus

$$\begin{aligned}\delta\mathfrak{v}(t,x,y) &= \mathfrak{v}(t,x+y) - \mathfrak{v}(t,x) - D\mathfrak{v}(t,x)y\\ &= (1-v)^+(t,x+y) - (1-v)(t,x) + Dv(t,x)y\\ &= (v-1)^+(t,x+y) - \delta v(t,x,y).\end{aligned}$$

Therefore, we have

$$\big(\delta\mathfrak{v}(t,x,y)\big)^- \ge \big(\delta v(t,x,y)\big)^+ - (v-1)^+(t,x+y),$$
$$\big(\delta\mathfrak{v}(t,x,y)\big)^+ \le \big(\delta v(t,x,y)\big)^- + (v-1)^+(t,x+y).$$

These imply

$$\begin{aligned}\mathfrak{v}_t - \mathcal{M}^-\mathfrak{v} &= -v_t - (2-\sigma)\int_{\mathbb{R}^d} \frac{\lambda(\delta\mathfrak{v}(t,x,y))^+ - \Lambda(\delta\mathfrak{v}(t,x,y))^-}{|y|^{d+\sigma}}\,dy\\ &\geq -v_t + \mathcal{M}^+v - (2-\sigma)(\Lambda+\lambda)\int_{\mathbb{R}^d}\frac{(v-1)^+(t,x+y)}{|y|^{d+\sigma}}\,dy.\end{aligned}$$

From Condition (iii) and an approximation, v satisfies

$$v_t - \mathcal{M}^+ v \leq 0.$$

On the other hand, we have

$$\int_{\mathbb{R}^d}\frac{(v-1)^+(t,x+y)}{|y|^{d+\sigma}}\,dy \leq \int_{\mathbb{R}^d}\frac{(P-1)^+(t,x+y)}{|y|^{d+\sigma}}\,dy.$$

Since P satisfies (3.9), for $(t,x)\in Q_{3/4}$, we have $P(t,x+y)\leq 1$ when $y\in B_{1/4}$, and thus the right-hand side above is equal to

$$\sum_{i=0}^{\infty}\int_{B_{\kappa^{i+1}-3/4}\setminus B_{\kappa^i-3/4}}\frac{(P-1)^+(t,x+y)}{|y|^{d+\sigma}}\,dy,$$

which by (3.9) is bounded by

$$\begin{aligned}&\sum_{i=0}^{\infty}\int_{B_{\kappa^{i+1}-3/4}\setminus B_{\kappa^i-3/4}}\frac{\kappa^{(i+1)\hat\alpha}-1}{|y|^{d+\sigma}}\,dy\\ &\leq C\sum_{i=0}^{\infty}(\kappa^{(i+1)\hat\alpha}-1)\kappa^{-i\sigma} = C\Big(\frac{\kappa^{\hat\alpha}}{1-\kappa^{\hat\alpha-\sigma}} - \frac{1}{1-\kappa^{-\sigma}}\Big)\leq C\big(\kappa^{\hat\alpha}-1\big),\end{aligned}$$

where C only depends on d,λ,Λ, and σ_0. Here we used the fact $\hat\alpha<\sigma/2$ and $\kappa\geq 2$ in the last inequality. The lemma is proved. $\square$

Let $\hat\theta = \lambda/(4\Lambda)$ and γ be the constant in Proposition 2.4. For any $r_1>0$, define the set

$$\mathcal{D}_{r_1} = \{(t,x)\in Q_{r_1} : v\geq 1-\hat\theta\}.$$

Lemma 3.5. *Suppose that $\hat\alpha\in(0,1/2)$ satisfying $\hat\alpha<\sigma/2$ and $\kappa\geq 2$. There exist some $\eta\in(0,1)$ sufficiently close to 1 and $c\in(0,1)$ sufficiently small, both depending only on d,λ,Λ, and σ, such that for $r_1 = c\kappa^{-\hat\alpha/\gamma}$,*

$$|\mathcal{D}_{r_1}|\leq \eta|Q_{r_1}|. \tag{3.11}$$

Proof. By contradiction we assume that $|\mathcal{D}_{r_1}|>\eta|Q_{r_1}|$, and consider

$$w := \int_{\mathbb{R}^d\setminus\mathbb{A}}\big(\delta u(t,x,y)-\delta u(0,0,y)\big)\frac{2-\sigma}{|y|^{d+\sigma}}\,dy.$$

By Condition (iii) and an approximation, w is a subsolution, i.e.,

$$w_t - \mathcal{M}^+ w\leq 0 \quad\text{in}\quad \mathbb{R}^{d+1}_0. \tag{3.12}$$

From (3.2) and (3.3), we know that in $\mathbb{R}^{d+1}_0$,

$$\frac{\lambda}{\Lambda}P - \frac{u_t-u_t(0,0)}{\Lambda}\leq N\leq \frac{\Lambda}{\lambda}P - \frac{u_t-u_t(0,0)}{\lambda}, \tag{3.13}$$

implying that in $Q_{1/2}$

$$\frac{\lambda}{\Lambda}P - \frac{[u_t]_{\gamma/\sigma,\gamma;Q_{1/2}}}{\Lambda}\big(|x|^\gamma + |t|^{\gamma/\sigma}\big)\leq N\leq \frac{\Lambda}{\lambda}P + \frac{[u_t]_{\gamma/\sigma,\gamma;Q_{1/2}}}{\lambda}\big(|x|^\gamma+|t|^{\gamma/\sigma}\big).$$

From (3.5), (3.9), and the above inequality, we obtain

$$\begin{aligned} w = P - v - N &\le \big(1 - \lambda/\Lambda\big)P - v + C\kappa^{\hat{\alpha}}\big(|x|^\gamma + |t|^{\gamma/\sigma}\big) \\ &\le 1 - \lambda/\Lambda - (1 - \hat{\theta}) + C\kappa^{\hat{\alpha}}\big(|x|^\gamma + |t|^{\gamma/\sigma}\big) \\ &\le -\lambda/\Lambda + \hat{\theta} + C\kappa^{\hat{\alpha}}\big(|x|^\gamma + |t|^{\gamma/\sigma}\big) \quad \text{in } \mathcal{D}_{r_1}, \end{aligned}$$

where C only depends on d, λ, Λ, and σ. Now we choose c sufficiently small depending only on d, λ, Λ, and σ, such that

$$\begin{aligned} &-\lambda/\Lambda + \hat{\theta} + C\kappa^{\hat{\alpha}}\big(|x|^\gamma + |t|^{\gamma/\sigma}\big) \\ &\le -3\hat{\theta} + C\kappa^{\hat{\alpha}}\big(c\kappa^{-\hat{\alpha}/\gamma}\big)^\gamma \le -3\hat{\theta} + Cc^\gamma \le -\hat{\theta} \quad \text{in } Q_{r_1}, \end{aligned} \tag{3.14}$$

which implies $w \le -\hat{\theta}$ in $\mathcal{D}_{r_1}$. Since w is a subsolution (3.12), it follows immediately that for any $\varepsilon \in (0, r_1)$, $\bar{w}_\varepsilon(t,x) := (w + \hat{\theta})^+(\varepsilon^\sigma t, \varepsilon x)$ is a subsolution as well. Moreover,

$$\Big|\{\bar{w}_\varepsilon \le 0\} \cap Q_{r_1/\varepsilon}\Big| > \eta \Big|Q_{r_1/\varepsilon}\Big|. \tag{3.15}$$

We estimate $\bar{w}_\varepsilon$ by applying Proposition 2.3 with $t_1 = -1$, $t_2 = 0$, and $\Omega = \mathbb{R}^d$

$$\begin{aligned} &\bar{w}_\varepsilon(0,0) \le C\int_{-1}^0 \int_{\mathbb{R}^d} \frac{\bar{w}_\varepsilon}{1 + |x|^{d+\sigma}}\,dx\,dt \\ &= C\int_{-1}^0 \int_{B_{r_1/\varepsilon}} \frac{\bar{w}_\varepsilon}{1 + |x|^{d+\sigma}}\,dx\,dt + C\int_{-1}^0 \int_{B^c_{r_1/\varepsilon}} \frac{\bar{w}_\varepsilon}{1 + |x|^{d+\sigma}}\,dx\,dt. \end{aligned} \tag{3.16}$$

We first consider the second term on the right-hand side of the inequality above

$$\begin{aligned} &\int_{-1}^0 \int_{B^c_{r_1/\varepsilon}} \frac{\bar{w}_\varepsilon}{1 + |x|^{d+\sigma}}\,dx\,dt \\ &\le \int_{-1}^0 \int_{B^c_{r_1/\varepsilon}} \frac{\hat{\theta}}{1 + |x|^{d+\sigma}}\,dx\,dt + \int_{-1}^0 \int_{B^c_{r_1/\varepsilon}} \frac{|w|(\varepsilon^\sigma t, \varepsilon x)}{1 + |x|^{d+\sigma}}\,dx\,dt \\ &\le C\hat{\theta}(\varepsilon/r_1)^\sigma + \int_{-\varepsilon^\sigma}^0 \int_{B^c_{r_1}} \frac{|w|(t,x)}{\varepsilon^{d+\sigma} + |x|^{d+\sigma}}\,dx\,dt. \end{aligned} \tag{3.17}$$

Since $|w| \le \max\{P, N\}$, from (3.9), for any $l \ge 0$,

$$\sup_{Q_{\kappa^l}} |w| \le \kappa^{\hat{\alpha} l}.$$

Therefore,

$$\begin{aligned} &\int_{-\varepsilon^\sigma}^0 \int_{B^c_{r_1}} \frac{|w|(t,x)}{\varepsilon^{d+\sigma} + |x|^{d+\sigma}}\,dx\,dt \\ &\le \int_{-\varepsilon^\sigma}^0 \int_{B_1 \setminus B_{r_1}} \frac{|w|}{\varepsilon^{d+\sigma} + |x|^{d+\sigma}}\,dx\,dt + \sum_{i=0}^\infty \int_{-\varepsilon^\sigma}^0 \int_{B_{\kappa^{i+1}} \setminus B_{\kappa^i}} \frac{|w|}{|x|^{d+\sigma}}\,dx\,dt \\ &\le C\big((\varepsilon/r_1)^\sigma + \varepsilon^\sigma \kappa^{\hat{\alpha}}\big), \end{aligned} \tag{3.18}$$

where C only depends on d, λ, Λ, and σ. We combine (3.17) and (3.18) to obtain that

$$C\int_{-1}^0 \int_{B^c_{r_1/\varepsilon}} \frac{\bar{w}_\varepsilon}{1 + |x|^{d+\sigma}}\,dx\,dt \le C\big((\varepsilon/r_1)^\sigma + \varepsilon^\sigma \kappa^{\hat{\alpha}}\big).$$

Recall that $r_1 = c\kappa^{-\hat{\alpha}/\gamma}$. For the right-hand side of the inequality above, we want to choose ε sufficiently small such that $C\varepsilon^\sigma(r_1^{-\sigma} + \kappa^{\hat{\alpha}}) \le \hat{\theta}/4$. Indeed, since $c \in (0,1)$ and $\sigma > \gamma$, we have $r_1^{-\sigma} \ge \kappa^{\hat{\alpha}}$. It is sufficient to fix ε such that $2C\varepsilon^\sigma r_1^{-\sigma} = \hat{\theta}/4$, i.e.,

$$\varepsilon/r_1 = \big(\hat{\theta}/(8C)\big)^{1/\sigma} := c_1,$$

where c_1 only depends on d, λ, Λ, and σ. In other words, by taking $\varepsilon = c_1 r_1$ we have

$$C\int_{-1}^0 \int_{B^c_{r_1/\varepsilon}} \frac{\bar{w}_\varepsilon}{1+|x|^{d+\sigma}}\,dx\,dt \le \hat{\theta}/4. \tag{3.19}$$

Next we estimate the first term on the right-hand side of (3.16) using (3.15):

$$\begin{aligned}
&C\int_{-1}^0 \int_{B_{1/c_1}} \frac{\bar{w}_\varepsilon}{1+|x|^{d+\sigma}}\,dx\,dt\\
&= C\int_{\left((-1,0)\times B_{1/c_1}\right)\cap\{\bar{w}_\varepsilon>0\}} \frac{\bar{w}_\varepsilon}{1+|x|^{d+\sigma}}\,dx\,dt\\
&\le C(1-\eta)|Q_{1/c_1}| \sup_{(-1,0)\times B_{1/c_1}} \bar{w}_\varepsilon\\
&\le C(1-\eta)|Q_{1/c_1}| \sup_{(-1,0)\times B_{r_1}} (w+\hat{\theta})^+ \le \hat{\theta}/4
\end{aligned} \tag{3.20}$$

upon taking η sufficiently close to 1 depending only on d, λ, Λ, and σ.

Combining (3.20) and (3.19) with (3.16), we have $\bar{w}_\varepsilon(0,0) \le \hat{\theta}/2$ indicating that $w(0,0) \le -\hat{\theta}/2$, which contradicts to $w(0,0) = 0$ by the definition of w. Therefore, the lemma is proved. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. We mainly focus on the case when $\sigma > 1$. At the end of the proof, we sketch the proof for the case when $\sigma = 1$. The proof of the case when $\sigma \in (0,1)$ is similar, and thus omitted.

Let η and c be the constants in Lemma 3.5 and $r_1 = c\kappa^{-\hat{\alpha}/\gamma}$. From (3.11),

$$\big|\{1 - v \ge \hat{\theta}\} \cap Q_{r_1}\big| \ge (1-\eta)\big|Q_{r_1}\big|.$$

Recall that $\tilde{Q}_{\delta r_1} = (-{r_1}^\sigma, -(\delta r_1)^\sigma) \times B_{r_1}$. For δ sufficiently small depending on η, we have

$$\big|\{1 - v \ge \hat{\theta}\} \cap \tilde{Q}_{\delta r_1}\big| \ge \frac{1-\eta}{2}\big|\tilde{Q}_{\delta r_1}\big|.$$

We set $r = \delta r_1/2$ and apply the weak Harnack inequality Corollary 2.2 to $(1-v)^+$ in $Q_{3/4}$ with (3.10) to obtain

$$\begin{aligned}
&\inf_{Q_r}(1-v) + C(\kappa^{\hat{\alpha}} - 1)r_1^\sigma\\
&\ge C(\delta)\|(1-v)^+\|_{L_{\varepsilon_1}(\tilde{Q}_{\delta r_1})} r_1^{-(d+\sigma)/\varepsilon_1}\\
&\ge C(\delta)\hat{\theta}\big((1-\eta)|\tilde{Q}_{\delta r_1}|/2\big)^{1/\varepsilon_1} r_1^{-(d+\sigma)/\varepsilon_1}\\
&= C(\delta)\hat{\theta}\big((1-\eta)(1-\delta^\sigma)/2\big)^{1/\varepsilon_1} =: 2\theta,
\end{aligned} \tag{3.21}$$

where θ is a small constant depending only on d, λ, Λ, and σ. We fix $\kappa = 4/(c\delta)^2$, where c is chosen according to (3.14), which guarantees that

$$\kappa^{-1} \le c\delta/2 \cdot \kappa^{-\hat{\alpha}/\gamma}$$

for any $\hat{\alpha} \in (0, \gamma/2)$. By the definitions of r above and r_1 in Lemma 3.5, the right-hand side of inequality above equals r, i.e., $\kappa^{-1} < r$.

Next, we choose $\hat{\alpha}_1 = \log(1+\theta/C)/\log\kappa$ such that for any $\hat{\alpha} \in (0, \hat{\alpha}_1)$,

$$C(\kappa^{\hat{\alpha}} - 1) r_1^\sigma < C(\kappa^{\hat{\alpha}} - 1) < \theta.$$

Therefore, from (3.21) and the inequality above we obtain that $\sup_{Q_r} v \le 1 - \theta$. Since $\kappa^{-1} < r$,

$$\sup_{Q_{\kappa^{-1}}} P \le \sup_{Q_r} P = P(t_0, x_0) = v(t_0, x_0) = \sup_{Q_r} v \le 1 - \theta. \tag{3.22}$$

Similarly,

$$\sup_{Q_{\kappa^{-1}}} N \le 1 - \theta. \tag{3.23}$$

Set

$$\hat{\alpha} = \min\big\{ -\log(1-\theta)/\log\kappa, \hat{\alpha}_1, \gamma/2 \big\}.$$

Then (3.22) and (3.23) implies

$$\sup_{Q_{\kappa^{-1}}} \kappa^{\hat{\alpha}} P \le 1, \quad \sup_{Q_{\kappa^{-1}}} \kappa^{\hat{\alpha}} N \le 1. \tag{3.24}$$

Let

$$\tilde{P}(t,x) = \kappa^{\hat{\alpha}} P(\kappa^{-1}x, \kappa^{-\sigma}t), \quad \tilde{N}(t,x) = \kappa^{\hat{\alpha}} N(\kappa^{-1}x, \kappa^{-\sigma}t),$$

and

$$\tilde{u}(t,x) = \kappa^{\sigma+\hat{\alpha}} u(\kappa^{-1}x, \kappa^{-\sigma}t).$$

From (3.13), we have

$$\frac{\lambda}{\Lambda}\tilde{P} - \frac{\tilde{u}_t - \tilde{u}_t(0,0))}{\Lambda} \le \tilde{N} \le \frac{\Lambda}{\lambda}\tilde{P} - \frac{\tilde{u}_t - \tilde{u}_t(0,0)}{\lambda}.$$

Since $\kappa \ge 2$ and $\hat{\alpha} \le \gamma$, we get

$$[\tilde{u}_t]_{\gamma/\sigma,\gamma;Q_{1/2}} \le \kappa^{\hat{\alpha}-\gamma}[u_t]_{\gamma/\sigma,\gamma;Q_{1/2}} \le [u_t]_{\gamma/\sigma,\gamma;Q_{1/2}}.$$

On the other hand, for any $l \ge 0$,

$$\sup_{Q_{\kappa^l}} \tilde{P} = \kappa^{\hat{\alpha}} \sup_{Q_{\kappa^{l-1}}} P \le \kappa^{\hat{\alpha}} \kappa^{(l-1)\hat{\alpha}} = \kappa^{l\hat{\alpha}}, \quad \sup_{Q_{\kappa^l}} \tilde{P} \le \kappa^{l\hat{\alpha}}.$$

Therefore, $\tilde{P}$ and $\tilde{N}$ satisfy all the conditions of P and N. Applying (3.24) to $\tilde{P}$ and $\tilde{N}$, we have

$$\sup_{Q_{\kappa^{-1}}} \tilde{P} \le \kappa^{-\hat{\alpha}}, \quad \sup_{Q_{\kappa^{-1}}} \tilde{N} \le \kappa^{-\hat{\alpha}},$$

which further implies that

$$\sup_{Q_{\kappa^{-2}}} P \le \kappa^{-2\hat{\alpha}}, \quad \sup_{Q_{\kappa^{-2}}} N \le \kappa^{-2\hat{\alpha}}.$$

By induction, for any $l \in \mathbb{N}$,

$$\sup_{Q_{\kappa^{-l}}} P \le \kappa^{-l\hat{\alpha}}, \quad \sup_{Q_{\kappa^{-l}}} N \le \kappa^{-l\hat{\alpha}}.$$

Therefore, we have in Q_1

$$P(t,x) \le C\big(|x|^{\hat{\alpha}} + |t|^{\hat{\alpha}/\sigma}\big), \quad N(t,x) \le C\big(|x|^{\hat{\alpha}} + |t|^{\hat{\alpha}/\sigma}\big).$$

Since for any $\eta \geq 1$, $\hat{u}(t,x) = \eta^{-\sigma-\alpha} u(\eta^\sigma t, \eta x)$ satisfies the same condition as u, replacing u by $\hat{u}$ in the definition of P and denoting it as $P_{\hat{u}}$, we obtain

$$P_{\hat{u}}(t,x) \leq C\big(|x|^{\hat{\alpha}} + |t|^{\hat{\alpha}/\sigma}\big) \quad \text{in } Q_1.$$

Returning to P, we have

$$\eta^{-\alpha} P(\eta^\sigma t, \eta x) \leq C\big(|x|^{\hat{\alpha}} + |t|^{\hat{\alpha}/\sigma}\big)$$

in Q_1, which further implies that

$$\sup_{Q_\eta} \frac{P(t,x)}{|x|^{\hat{\alpha}} + |t|^{\hat{\alpha}/\sigma}} \leq C\eta^{\alpha-\hat{\alpha}}.$$

Let $\eta \to \infty$ yields

$$\sup_{(t,x)\in\mathbb{R}^{d+1}_0} \frac{P(t,x)}{|x|^{\hat{\alpha}} + |t|^{\hat{\alpha}/\sigma}} = 0,$$

which gives $P = 0$. Similarly, $N = 0$.

From the definition of P and N, we have

$$u(t,x+y) - u(t,x) - Du(t,x)y = u(0,y) - u(0,0) - Du(0,0)y.$$

Taking derivative in y, we have for any $t \in (-\infty, 0)$ and $x, y \in \mathbb{R}^d$

$$Du(t,x+y) - Du(t,x) = Du(0,y) - Du(0,0),$$

which implies for fixed t, u is a polynomial in x of order at most two. Using Condition (i) with $\beta = 0$, we infer that this order is at most ν. Using Condition (ii) and $P = N = 0$, we get $u_t = c$ for some constant c. The proof is completed for $\sigma > 1$.

Finally, we sketch the proof for $\sigma = 1$. From Condition (ii), similar to the proof above, we know that $u(\cdot + s, \cdot + h) - u$, and thus Du and u_t are both sub and supersolutions, and are in $C^{\gamma/\sigma,\gamma}(Q_{1/2})$. By Proposition 2.4, we have

$$[u_t]_{\gamma/\sigma,\gamma;Q_{1/2}} + [Du]_{\gamma/\sigma,\gamma;Q_{1/2}} \leq C \sup_{t\in[0,1]} \int_{\mathbb{R}^d} \frac{|Du| + |u_t|}{1 + |x|^{d+1}}\, dx.$$

From Condition (i), the right-hand side of the inequality above is less than

$$\int_{B_1} \frac{1}{1+|x|^{1+d}}\, dx + \sum_{j=1}^{\infty} \int_{B_{2^j}\setminus B_{2^{j-1}}} \frac{C2^{j\alpha}}{1+|x|^{1+d}}\, dx \leq \frac{C}{1-2^{\alpha-1}}$$

for any $\alpha \in (0,1)$. By taking $\hat{\alpha} \leq \gamma$ and scaling as before, we can prove that

$$[u]_{1+\gamma,1+\gamma;Q_R} \leq CR^{\alpha-\gamma},$$

i.e., u must be a linear function by sending $R \to \infty$. The theorem is proved. $\square$

4. Schauder estimate for nonlocal parabolic equations

In this section, we prove Theorem 1.1 by applying the Liouville theorem, a blow-up analysis, and a localization procedure. In the rest of the paper, we do not specify the domain associated with the norm when it is $\mathbb{R}^{d+1}_0 = (-\infty, 0) \times \mathbb{R}^d$.

4.1. **Equations with translation invariant kernels.** In this subsection, we consider equations with translation invariant kernels, i.e., $K = K(y)$. The main result of the subsection is the following theorem.

Theorem 4.1. *Let $\sigma \in (0,2)$ be a constant and $\mathcal{A}$ be an index set. There exists a constant $\hat{\alpha} > 0$ depending on d, λ, Λ, and σ, such that given $0 < \alpha' < \alpha < \hat{\alpha}$ satisfying $[\sigma + \alpha] < \sigma + \alpha' < \sigma + \alpha$ the following holds. Let $u \in C^{1+\alpha'/\sigma, \alpha'+\sigma}\big((-1,0) \times \mathbb{R}^d\big) \cap C^{1+\alpha/\sigma,\alpha+\sigma}(Q_1)$ satisfy*

$$u_t = \inf_{a \in \mathcal{A}} \big(L_a u + f_a(t,x)\big) \quad \text{in } Q_1,$$

where $L_a \in \mathcal{L}_0(\sigma, \lambda, \Lambda)$ with $K_a = K_a(y)$ for any $a \in \mathcal{A}$. Assume that

$$\sup_{(t,x) \in Q_1} \Big| \inf_{a \in \mathcal{A}} f_a(t,x) \Big| < \infty.$$

Then

$$[u]_{1+\alpha/\sigma, \alpha+\sigma; Q_{1/2}} \le C [u]_{1+\alpha'/\sigma, \alpha'+\sigma; (-1,0)\times\mathbb{R}^d} + C \sup_{a \in \mathcal{A}} [f_a]_{\alpha/\sigma, \alpha; Q_1},$$

where C only depends on $d, \lambda, \Lambda, \sigma, \alpha$, and α'.

We denote

$$Q^k = \big(-1 + 2^{-(k+1)\sigma}/(1 - 2^{-\sigma}), 0\big) \times B_{1-2^{-k}}(0) \tag{4.1}$$

for all sufficiently large integers k such that $2^{-(k+1)\sigma} < 1 - 2^{-\sigma}$. We shall prove a stronger result:

$$\begin{aligned} &\sup_k 2^{-k(\alpha-\alpha')} [u]_{1+\alpha/\sigma, \alpha+\sigma; Q^k} \\ &\le C [u]_{1+\alpha'/\sigma, \alpha'+\sigma; (-1,0)\times\mathbb{R}^d} + C \sup_{a \in \mathcal{A}} [f_a]_{\alpha/\sigma, \alpha; Q_1}. \end{aligned} \tag{4.2}$$

The conclusion of the theorem is a particular case for k large only depending on σ so that $Q_{1/2} \subset Q^k$. Since we assume that $u \in C^{1+\alpha/\sigma,\alpha+\sigma}(Q_1)$, there exists an integer k such that

$$2^{-k(\alpha-\alpha')} [u]_{1+\alpha/\sigma,\alpha+\sigma;Q^k} = \sup_l 2^{-l(\alpha-\alpha')} [u]_{1+\alpha/\sigma,\alpha+\sigma;Q^l}.$$

Next, we prove (4.2) by contradiction. Assume that we can find solutions u_j and index sets $\mathcal{A}_j$ such that

$$\begin{aligned} &\partial_t u_j = \inf_{a \in \mathcal{A}_j} (L_a u_j + f_a) \quad \text{in} \quad Q_1, \qquad \sup_{(t,x)\in Q_1} \Big| \inf_{a \in \mathcal{A}_j} f_a(t,x) \Big| < \infty, \\ &[u_j]_{1+\alpha'/\sigma, \sigma+\alpha'; (-1,0)\times\mathbb{R}^d} + \sup_{a \in \mathcal{A}_j} [f_a]_{\alpha/\sigma,\alpha;Q_1} \le 1, \\ &\text{and} \quad \sup_k 2^{-k(\alpha-\alpha')} [u_j]_{1+\alpha/\sigma,\sigma+\alpha;Q^k} \ge j, \end{aligned} \tag{4.3}$$

where for any $a \in \mathcal{A}_k$, $L_a \in \mathcal{L}_0$ with $K_a = K_a(y)$. As explained above, for each j there exists an integer k_j so that

$$2^{-k_j(\alpha-\alpha')} [u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}} = \sup_k 2^{-k(\alpha-\alpha')} [u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^k}.$$

Lemma 4.2. *For any $j \geq 1$, we have*

$$[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}} \leq \sup_{r>0} \sup_{(t,x)\in Q^{k_j}} r^{-(\alpha-\alpha')}[u_j]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_r(t,x)\cap\{t>-1\}}$$
$$\leq 4^{\alpha-\alpha'}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}. \tag{4.4}$$

Moreover, we can find $(t_j, x_j) \in Q^{k_j}$ and r_j such that

$$\frac{1}{2}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}} \leq r_j^{-(\alpha-\alpha')}[u_j]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_{r_j}(t_j,x_j)\cap\{t>-1\}} \tag{4.5}$$

and

$$2^{k_j} r_j \to 0 \quad \text{as } j \to \infty. \tag{4.6}$$

Proof. The first inequality in (4.4) follows from the fact that for any $(t,x),(s,y) \in Q^{k_j}$ with $t \geq s$, we have $(s,y) \in Q_r(t,x) \cap \{t > -1\}$, where $r = \max(|x-y|, |t-s|^{1/\sigma})$. See, for instance, Claim 3.2 of [23]. For the second inequality, if $r \leq 2^{-(k_j+1)}$, for any $(t,x) \in Q^{k_j}$, we have $Q_r(t,x) \subset Q^{k_j+1}$ and

$$r^{-(\alpha-\alpha')}[u_j]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_r(t,x)\cap\{t>-1\}}$$
$$\leq 2^{\alpha-\alpha'}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j+1}} \leq 4^{\alpha-\alpha'}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}},$$

where the last inequality is due to the choice of k_j. On the other hand, if $r > 2^{-(k_j+1)}$, for any $(t,x) \in Q^{k_j}$,

$$r^{-(\alpha-\alpha')}[u_j]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_r(t,x)\cap\{t>-1\}}$$
$$\leq 2^{(k_j+1)(\alpha-\alpha')}[u_j]_{1+\alpha'/\sigma,\alpha'+\sigma;(-1,0)\times\mathbb{R}^d} \leq 2^{\alpha-\alpha'}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}},$$

where the last inequality follows from (4.3). Thus, we obtain the second inequality in (4.4).

Due to (4.4), we can find $(t_j, x_j) \in Q^{k_j}$ and r_j such that (4.5) is satisfied and thus by (4.3),

$$(2^{k_j} r_j)^{\alpha-\alpha'} \leq \frac{2[u_j]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_{r_j}(t_j,x_j)\cap\{t>-1\}}}{2^{-k_j(\alpha-\alpha')}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \to 0 \quad \text{as } j \to \infty,$$

which further implies (4.6). The lemma is proved. □

Let T_j be the Taylor expansion of u_j at $X_j = (t_j, x_j)$ of order $\nu = [\sigma + \alpha]$ in x and 1 in t. Now we consider the blow-up sequence

$$v_j(t,x) = \frac{u_j(t_j + r_j^\sigma t, x_j + r_j x) - T_j(t_j + r_j^\sigma t, x_j + r_j x)}{r_j^{\sigma+\alpha}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}}.$$

Here (t_j, x_j) and r_j are from Lemma 4.2. Note that v_j is well defined on $(-R_j^\sigma, 0) \times \mathbb{R}^d$, where by Lemma 4.2,

$$R_j := 2^{-(k_j+1)} r_j^{-1} \to \infty \quad \text{as} \quad j \to \infty.$$

Observe that from (4.5) and (4.6), for sufficiently large j such that $r_j \leq 2^{-(k_j+1)}$,

$$[v_j]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_1} = \frac{r_j^{\sigma+\alpha'}[u_j]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_{r_j}(t_j,x_j)}}{r_j^{\sigma+\alpha}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \geq 1/2. \tag{4.7}$$

Lemma 4.3. *For any $R > 0$ and $\beta \in [0, \sigma + \alpha']$, we have*

$$[v_j]_{\beta/\sigma,\beta;Q_R\cap\{t>-R_j^\sigma\}} \le CR^{\sigma+\alpha-\beta}, \tag{4.8}$$

where C depends only on α and α'. Moreover, for any $0 < R < R_j$ and $\beta \in [0, \sigma + \alpha]$, we have

$$[v_j]_{\beta/\sigma,\beta;Q_R} \le CR^{\sigma+\alpha-\beta}, \tag{4.9}$$

where C depends only on α and α'. Thus, we can find $v \in C^{1+\alpha/\sigma,\sigma+\alpha}(\mathbb{R}^{d+1}_0)$ such that v satisfies (4.9) for any $R > 0$ and $\beta \in [0, \sigma + \alpha]$, and along a subsequence $v_j \to v$ in $C^{1+\beta/\sigma,\sigma+\beta}$ locally uniformly for any $\beta \in [0, \sigma + \alpha)$.

We remark that (4.8) will be used below to prove that v satisfies Condition (iii) in Theorem 3.1, and (4.9) will be used to show that v satisfies Condition (i).

Proof of Lemma 4.3. For any $R > 0$ and $\beta \in [0, \sigma + \alpha']$,

$$\begin{aligned}
&[v_j]_{\beta/\sigma,\beta;Q_R\cap\{t>-R_j^\sigma\}}\\
&= \frac{[u_j(t_j + r_j^\sigma\cdot, x_j + r_j\cdot) - T_j(t_j + r_j^\sigma\cdot, x_j + r_j\cdot)]_{\beta/\sigma,\beta;Q_R\cap\{t>-R_j^\sigma\}}}{r_j^{\sigma+\alpha}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}}\\
&\le \frac{r_j^\beta[u_j - T_j]_{\beta/\sigma,\beta;Q_{Rr_j}(t_j,x_j)\cap\{t>-1\}}}{r_j^{\sigma+\alpha}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}}\\
&\le \frac{r_j^\beta(Rr_j)^{\sigma+\alpha'-\beta}[u_j]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_{Rr_j}(t_j,x_j)\cap\{t>-1\}}}{r_j^{\sigma+\alpha}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \le CR^{\sigma+\alpha-\beta},
\end{aligned}$$

where we used (4.4) in the last inequality.

For any $R < R_j$, by the choice of k_j we have

$$\begin{aligned}
[v_j]_{1+\alpha/\sigma,\alpha+\sigma;Q_R} &= \frac{[u_j(t_j + r_j^\sigma\cdot, x_j + r_j\cdot)]_{1+\alpha/\sigma,\alpha+\sigma;Q_R}}{r_j^{\sigma+\alpha}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}}\\
&= \frac{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q_{Rr_j}(t_j,x_j)}}{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \le \frac{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j+1}}}{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \le 2^{\alpha-\alpha'}.
\end{aligned}$$

Using the interpolation inequality, we reach (4.9). The last statement of the lemma follows from the Arzela-Ascoli theorem and the Cauchy diagonal method. □

Lemma 4.4. *The function v in Lemma 4.3 satisfies the conditions in Theorem 3.1.*

Proof. By Lemma 4.3, Condition (i) is satisfied. Next we verify Condition (iii) for $\sigma \in (1, 2)$. For any measure μ with compact support and $\delta \in (0, 1)$, we define

$$V_j(t, x) = \int_{\mathbb{R}^d} v_j(t, x + h) - v_j(t, x) - \frac{v_j(t, x) - v_j(t, x - \delta h)}{\delta}\, d\mu(h).$$

Since T_j is linear in t, from the definition of v_j, we have

$$\partial_t V_j(t,x) = \frac{r_j^{-\alpha}}{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}}$$
$$\cdot \int_{\mathbb{R}^d} \Big[\partial_t u_j(t_j + r_j^\sigma t, x_j + r_j(x+h)) - \partial_t u_j(t_j + r_j^\sigma t, x_j + r_j x)$$
$$- \frac{\partial_t u_j(t_j + r_j^\sigma t, x_j + r_j x) - \partial_t u_j(t_j + r_j^\sigma t, x_j + r_j(x-\delta h))}{\delta}\Big]\, d\mu(h),$$

which is equal to

$$\frac{r_j^{-\alpha}}{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \Big[\int_{\mathbb{R}^d} \partial_t u_j(t_j + r_j^\sigma t, x_j + r_j(x+h))\, d\mu(h)$$
$$+ \int_{\mathbb{R}^d} \frac{\partial_t u_j(t_j + r_j^\sigma t, x_j + r_j(x-\delta h))}{\delta}\, d\mu(h)$$
$$- (1+1/\delta)\|\mu\|_{L_1} \partial_t u_j(t_j + r_j^\sigma t, x_j + r_j x)\Big]. \tag{4.10}$$

For any $a \in \mathcal{A}_j$, define $\hat{K}_a(y) = r_j^{d+\sigma} K_a(r_j y)$, which satisfies

$$\frac{\lambda}{|y|^{d+\sigma}} \le \hat{K}_a(y) \le \frac{\Lambda}{|y|^{d+\sigma}},$$

and $\hat{L}_a$ be the corresponding operator with kernel $\hat{K}_a$.

Clearly,

$$\hat{L}_a V_j = \int_{\mathbb{R}^d} \Big[\hat{L}_a(v_j(t,x+h) - v_j(t,x)) - \frac{\hat{L}_a(v_j(t,x) - v_j(t,x-\delta h))}{\delta}\Big] d\mu(h)$$
$$= \frac{r_j^{-\alpha}}{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \int_{\mathbb{R}^d} \Big\{(L_a u_j)(t_j + r_j^\sigma t, x_j + r_j x + r_j h)$$
$$- (L_a u_j)(t_j + r_j^\sigma t, x_j + r_j x)$$
$$- \frac{(L_a u_j)(t_j + r_j^\sigma t, x_j + r_j x) - (L_a u_j)(t_j + r_j^\sigma t, x_j + r_j(x-\delta h))}{\delta}\Big\}\, d\mu(h).$$

where in the second equality, we used the definitions of $\hat{L}_a$, v_j and the fact that since T_j is at most second-order in x variable, for $\sigma > 1$ and any $y \in \mathbb{R}^d$

$$\delta T_j(t, x+h, y) - \delta T_j(t,x,y) = 0.$$

Therefore, for any $(t,x) \in (-R_j^\sigma, 0) \times \mathbb{R}^d$,

$$\sup_{a\in\mathcal{A}_j} \hat{L}_a(V_j) = \frac{r_j^{-\alpha}}{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \sup_{a\in\mathcal{A}_j} \Big\{ \int_{\mathbb{R}^d} \Big((L_a u_j)(t_j + r_j^\sigma t, x_j + r_j x + r_j h)$$
$$+ \frac{(L_a u_j)(t_j + r_j^\sigma t, x_j + r_j(x-\delta h))}{\delta}\Big) d\mu(h) - (1 + \frac{1}{\delta})(L_a u_j)(t_j + r_j^\sigma t, x_j + r_j x)\Big\}$$

$$
\begin{aligned}
&= \frac{r_j^{-\alpha}}{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \sup_{a\in\mathcal{A}_j} \Big\{ \int_{\mathbb{R}^d} (L_a u_j)(t_j + r_j^\sigma t, x_j + r_j x + r_j h) \\
&\quad + f_a(t_j + r_j^\sigma r, x_j + r_j x + r_j h) + \frac{1}{\delta}\big((L_a u_j)(t_j + r_j^\sigma t, x_j + r_j(x - \delta h)) \\
&\quad + f_a(t_j + r_j^\sigma t, x_j + r_j(x-\delta h))\big)\, d\mu(h) \\
&\quad - (1+\frac{1}{\delta})((L_a u_j)(t_j + r_j^\sigma t, x_j + r_j x) + f_a(t_j + r_j^\sigma t, x_j + r_j x)) \cdot \|\mu\|_{L_1} \\
&\quad - \int_{\mathbb{R}^d} \Big(f_a(t_j + r_j^\sigma t, x_j + r_j x + r_j h) - f_a(t_j + r_j^\sigma t, x_j + r_j x) \\
&\quad + \frac{1}{\delta}(f_a(t_j + r_j^\sigma t, x_j + r_j(x-\delta h)) - f_a(t_j + r_j^\sigma t, x_j + r_j x)\Big)\, d\mu(h) \Big\}.
\end{aligned}
$$

Note that for sufficiently large j such that $\max((-t)^{1/\sigma}, |x| + |h|) \le R_j$ whenever $h \in \mathrm{supp}\mu$, we have

$$
\begin{aligned}
&|f_a(t_j + r_j^\sigma t, x_j + r_j x + r_j h) - f_a(t_j + r_j^\sigma t, x_j + r_j x)| \le [f_a]_{\alpha/\sigma,\alpha;Q_1} |r_j h|^\alpha, \\
&|f_a(t_j + r_j^\sigma t, x_j + r_j(x-\delta h)) - f_a(t_j + r_j^\sigma t, x_j + r_j x)| \le [f_a]_{\alpha/\sigma,\alpha;Q_1} |\delta r_j h|^\alpha.
\end{aligned}
$$

Therefore, by the inequality

$$
\sup\{f + g - h\} \ge \inf f + \inf g - \inf h,
$$

we have that for $(t,x) \in \mathbb{R}_0^{d+1}$ and $h \in \mathrm{supp}\,\mu$ so that $\max((-t)^{1/\sigma}, |x|+|h|) \le R_j$,

$$
\begin{aligned}
&\sup_{a\in\mathcal{A}_j} \hat{L}_a(V_j)(t,x) \\
&\ge \frac{r_j^{-\alpha}}{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \Big[\inf_{a\in\mathcal{A}_j} \int_{\mathbb{R}^d} (L_a u_j)(t_j + r_j^\sigma t, x_j + r_j x + r_j h) \\
&\quad + f_a(t_j + r_j^\sigma r, x_j + r_j x + r_j h)\, d\mu(h) \\
&\quad + \inf_{a\in\mathcal{A}_j} \int_{\mathbb{R}^d} \frac{1}{\delta}\Big((L_a u_j)(t_j + r_j^\sigma t, x_j + r_j(x-\delta h)) \\
&\quad + f_a(t_j + r_j^\sigma t, x_j + r_j(x-\delta h))\Big)\, d\mu(h) \\
&\quad - \frac{\delta+1}{\delta}\|\mu\|_{L_1} \inf_a [(L_a u_j)(t_j + r_j^\sigma t, x_j + r_j x) + f_a(t_j + r_j^\sigma t, x_j + r_j x)] \\
&\quad - (1+\delta^{\alpha-1}) r_j^\alpha \sup_{a\in\mathcal{A}_j} [f_a]_{\alpha/\sigma,\alpha;Q_1} \int_{\mathbb{R}^d} |h|^\alpha\, d\mu(h)\Big].
\end{aligned}
\tag{4.11}
$$

Since each u_j satisfies

$$
\partial_t u_j = \inf_{a\in\mathcal{A}_j} (L_a u_j + f_a) \quad \text{in } Q_1, \tag{4.12}
$$

it follows from (4.10) and (4.11) that in any bounded subset of $\mathbb{R}_0^{d+1}$, for sufficiently large j,

$$
\partial_t V_j - \sup_{a\in\mathcal{A}_j} \hat{L}_a V_j \le \frac{1+\delta^{\alpha-1}}{[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}}} \sup_{a\in\mathcal{A}_j} [f_a]_{\alpha/\sigma,\alpha} \int_{\mathbb{R}^d} |h|^\alpha\, d\mu(h). \tag{4.13}
$$

We denote

$$
V(t,x) := \int_{\mathbb{R}^d} \Big[v(t,x+h) - v(t,x) - \frac{v(t,x) - v(t,x-\delta h)}{\delta} \Big]\, d\mu(h).
$$

For fixed $(t,x) \in \mathbb{R}^{d+1}_0$, by (4.8) in Lemma 4.3 and using the fact that μ has compact support, we have

$$\lim_j \partial_t V_j(t,x) = \partial_t \lim_j V_j(t,x) = \partial_t V(t,x), \tag{4.14}$$

for $|y| \le 1$,

$$|\delta V_j(t,x,y)| \le C|y|^{\sigma+\alpha'}, \quad |\delta V(t,x,y)| \le C|y|^{\sigma+\alpha'}, \tag{4.15}$$

and for $|y| > 1$,

$$|V_j(t,y)|, |V(t,y)| \le C|y|^{\alpha}, \tag{4.16}$$

where C depends on μ. Clearly,

$$\sup_{a\in\mathcal{A}_j} \big|L_a(V_j - V)(t,x)\big| \le C\int_{\mathbb{R}^d} \big|\delta(V_j-V)(t,x,y)\big||y|^{-d-\sigma}\,dy.$$

It follows from Lemma 4.3 that $\delta(V_j - V) \to 0$ locally uniformly. Therefore, by (4.15), (4.16), and the dominated convergence theorem, we have

$$\lim_j \sup_{a\in\mathcal{A}_j} \big|(L_a(V_j - V)(t,x))\big| = 0,$$

i.e.,

$$\lim_j \sup_{a\in\mathcal{A}_k} L_a V_j(t,x) = \sup_{a\in\mathcal{A}_k} L_a V(t,x). \tag{4.17}$$

Since μ has compact support, by Lemma 4.2 and (4.3), we have $R_j \to \infty$ and

$$[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}} \ge 2^{-k_j(\alpha-\alpha')}[u_j]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}} \to \infty.$$

For fixed $\delta \in (0,1)$, we send j to infinity to get from (4.13), (4.14), and (4.17) that

$$\partial_t V - \mathcal{M}^+ V \le 0 \quad \text{in } \mathbb{R}^{d+1}_0.$$

By sending δ to 0 and using the dominated convergence theorem, we conclude that

$$\int_{\mathbb{R}^d} \big(v(t,x+h) - v(t,x) - h^T Dv(t,x)\big)\,d\mu(h)$$

is a subsolution as well. Therefore, for $\sigma > 1$, v satisfies Condition (iii).

It remains to verify that v satisfies Condition (ii). Clearly, for fixed $(t,x),(s,h) \in \mathbb{R}^{d+1}_0$, when j is sufficiently large,

$$\begin{aligned}\partial_t\big(v_j(t+s,x+h) - v_j(t,x)\big) &= r_j^{-\alpha}[u_j]^{-1}_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}} \\ &\cdot \big(\partial_t u_j(t_j + r_j^\sigma(t+s), x_j + r_j(x+h)) - \partial_t u_j(t_j + r_j^\sigma t, x_j + r_j x)\big).\end{aligned} \tag{4.18}$$

On the other hand,

$$\begin{aligned}\mathcal{M}^-\big(v_j(t+s,x+h) - v_j(t,x)\big) &= r_j^{-\alpha}[u_j]^{-1}_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}} \\ &\cdot \mathcal{M}^-\big(u_j(t_j + r_j^\sigma(t+s), x_j + r_j(x+h)) - u_j(t_j + r_j^\sigma t, x_j + r_j x)\big).\end{aligned} \tag{4.19}$$

Combining (4.12), (4.18), and (4.19), we obtain that for j sufficiently large,

$$\begin{aligned}&\partial_t\big(v_j(t+s,x+h) - v_j(t,x)\big) - \mathcal{M}^-\big(v_j(t+s,x+h) - v_j(t,x)\big) \\ &\ge -[u_j]^{-1}_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_j}} \sup_a [f_a]_{\alpha/\sigma,\alpha;Q_1}.\end{aligned}$$

By sending j to infinity, we get for any $(t,x) \in \mathbb{R}^{d+1}_0$,

$$\partial_t\big(v(t+s,x+h) - v(t,x)\big) - \mathcal{M}^-\big(v(t+s,x+h) - v(t,x)\big) \ge 0.$$

Similarly,

$$\partial_t\big(v(t+s,x+h)-v(t,x)\big)-\mathcal{M}^+\big(v(t+s,x+h)-v(t,x)\big)\le 0.$$

The lemma is proved. □

Now we are ready to finish

Proof of Theorem 4.1. By Lemma 4.4 and Theorem 3.1, v is a polynomial of order ν in x and 1 in t. Since at the origin v_j along with its first derivative in t and up to ν-th order derivatives in x are 0, by Lemma 4.3 the same is true for v. Therefore, $v\equiv 0$. This gives us a contradiction with (4.7) and Lemma 4.3. The proof is completed. □

4.2. **Equations with (t,x)-dependent kernels.** In this subsection, we consider the case that kernels also depend on (t,x) and Hölder continuous in (t,x), i.e., there exists $A>0$ such that for any $a\in\mathcal{A}$, (1.4) is satisfied. We only prove Theorem 1.1 in the case when $\sigma+\alpha>2$ and the proof of the cases $\sigma+\alpha<2$ is similar and actually simpler. We divide the proof into several steps.

Let η be a nonnegative smooth cutoff function with $\eta\equiv 1$ in Q_1 and vanishes outside $(-(5/4)^\sigma,(5/4)^\sigma)\times B_{5/4}$. Set $v:=\eta u\in C^{1+\alpha/\sigma,\alpha+\sigma}$ and note that in Q_1,

$$\begin{aligned}
v_t=\eta u_t+\eta_t u&=\inf_{a\in\mathcal{A}}(\eta L_a u+\eta f_a+\eta_t u)\\
&=\inf_{a\in\mathcal{A}}(L_a v+h_a+\eta f_a+\eta_t u)\\
&=\inf_{a\in\mathcal{A}}\Big(\int_{\mathbb{R}^d}\delta v(t,x,y)K_a(0,0,y)\,dy+g_a+h_a+\eta f_a+\eta_t u\Big),
\end{aligned}$$

where

$$\begin{aligned}
h_a&=\eta L_a u-L_a v\\
&=\int_{\mathbb{R}^d}\big((\eta(t,x)-\eta(t,x+y))u(t,x+y)+y^T D\eta(t,x)u(t,x)\big)K_a(t,x,y)\,dy
\end{aligned}$$

and

$$g_a=\int_{\mathbb{R}^d}\delta v(t,x,y)\big(K_a(t,x,y)-K_a(0,0,y)\big)\,dy.$$

Here in order to apply the argument of freezing the coefficients, we subtracted and added $K_a(0,0,y)$ in the formula above.

Lemma 4.5. *Assume that $u\in C^{1+\alpha/\sigma,\sigma+\alpha}(Q_{11/8})\cap C^{\alpha/\sigma,\alpha}((-(11/8)^\sigma,0)\times\mathbb{R}^d)$. Let h_a and g_a be functions defined above. Then for any $\alpha\in\mathcal{A}$, we have*

$$[g_a]_{\alpha/\sigma,\alpha;Q_1}\le CA\big([v]_{1+\alpha/\sigma,\alpha+\sigma}+[v]_{\alpha/\sigma,\alpha}\big),\tag{4.20}$$

$$[h_a]_{\alpha/\sigma,\alpha;Q_1}\le C(A+1)\big(\|u\|_{\alpha/\sigma,\alpha;(-(11/8)^\sigma,0)\times\mathbb{R}^d}+\|D^2u\|_{L_\infty(Q_{11/8})}\big).\tag{4.21}$$

Proof. For $(t,x),(t',x')\in Q_1$, set $l=\max(|x-x'|,|t-t'|^\sigma)$. Without loss of generality, we may assume that $l\le 1/4$.

Estimates of g_a: From the definition and the triangle inequality,

$$\begin{aligned}
&\big|g_a(t,x) - g_a(t',x')\big| \\
&= \Big| \int_{\mathbb{R}^d} \delta v(t,x,y)\big(K_a(t,x,y) - K_a(0,0,y)\big)\,dy \\
&\quad - \int_{\mathbb{R}^d} \delta v(t',x',y)\big(K_a(t',x',y) - K_a(0,0,y)\big)\,dy\Big| \\
&\le \Big| \int_{\mathbb{R}^d} \big(\delta v(t,x,y) - \delta v(t',x',y)\big)\big(K_a(t,x,y) - K_a(0,0,y)\big)\,dy\Big| \\
&\quad + \Big| \int_{\mathbb{R}^d} \delta v(t',x',y)\big(K_a(t,x,y) - K_a(t',x',y)\big)\,dy\Big| \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Then we estimate I and II separately. First, similar to (3.8), I is less than

$$\begin{aligned}
&\int_{B_l} \Big|\big(\delta v(t,x,y) - \delta v(t',x',y)\big)\big(K_a(t,x,y) - K_a(0,0,y)\big)\Big|\,dy \\
&\quad + \int_{\mathbb{R}^d\setminus B_l} \Big|\big(\delta v(t,x,y) - \delta v(t',x',y)\big)\big(K_a(t,x,y) - K_a(0,0,y)\big)\Big|\,dy := \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

Applying (3.6), we have

$$\begin{aligned}
\mathrm{I}_1 &\le C[v]_{1+\alpha/\sigma,\alpha+\sigma;Q_{5/4}} \int_{B_l} l^{\alpha+\sigma-2}|y|^2\big(K_a(t,x,y) - K_a(0,0,y)\big)\,dy \\
&\le AC[v]_{1+\alpha/\sigma,\alpha+\sigma} l^{\alpha+\sigma-2}\big(|x|^\alpha + |t|^{\alpha/\sigma}\big) \int_{B_l} |y|^2|y|^{-d-\sigma}\,dy \\
&= CAl^\alpha[v]_{1+\alpha/\sigma,\alpha+\sigma}.
\end{aligned}$$

For I_2, we have

$$\begin{aligned}
\mathrm{I}_2 &\le C[v]_{1+\alpha/\sigma,\alpha+\sigma} l \int_{\mathbb{R}^d\setminus B_l} |y|^{\sigma+\alpha-1}\big|K_a(t,x,y) - K_a(0,0,y)\big|\,dy \\
&\le CA[v]_{1+\alpha/\sigma,\alpha+\sigma} l^\alpha\big(|x|^\alpha + |t|^{\alpha/\sigma}\big) \le CAl^\alpha[v]_{1+\alpha/\sigma,\alpha+\sigma}.
\end{aligned}$$

Next, we bound

$$\begin{aligned}
\mathrm{II} &\le \int_{\mathbb{R}^d\setminus B_1} \big([v]_{\alpha/\sigma,\alpha}|y|^\alpha + \|Dv\|_{L_\infty}|y|\big)\big|K_a(t,x,y) - K_a(t',x',y)\big|\,dy \\
&\quad + \int_{B_1} \|D^2v\|_{L_\infty}|y|^2\big|K_a(t,x,y) - K_a(t',x',y)\big|\,dy \\
&\le CAl^\alpha\big([v]_{\alpha/\sigma,\alpha} + \|Dv\|_\infty + \|D^2v\|_{L_\infty}\big).
\end{aligned}$$

Combining the estimates of I, II, and the interpolation inequality, we get

$$|g_a(t,x) - g_a(t',x')| \le CAl^\alpha\big([v]_{1+\alpha/\sigma,\alpha+\sigma} + [v]_{\alpha/\sigma,\alpha}\big),$$

so we obtain (4.20).

Estimates of h_a: For simplicity of notation, we denote

$$\xi(t,x,y) = \big(\eta(t,x) - \eta(t,x+y)\big)u(t,x+y) + y^T D\eta(t,x)u(t,x). \tag{4.22}$$

By the Leibniz rule, we have

$$\begin{aligned}\xi(t,x,y) &= y^T \int_0^1 \big(D\eta(t,x)u(t,x) - D\eta(t,x+sy)u(t,x+y)\big)\,ds \\ &= -\int_0^1\int_0^1 \big(u(t,x)sy^T D^2\eta(t,x+s'sy)y \\ &\quad + y^T D\eta(t,x+sy)y^T Du(t,x+s'y)\big)\,ds'\,ds,\end{aligned} \tag{4.23}$$

which implies that when $|y|\le 1/8$,

$$|\xi(t,x,y)| \le C|y|^2\big(\|u\|_{L_\infty(Q_{11/8})} + \|Du\|_{L_\infty(Q_{11/8})}\big). \tag{4.24}$$

On the other hand, clearly when $|y|\ge 1/8$,

$$|\xi(t,x,y)| \le C\big(\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} + |y|\|u\|_{L_\infty(Q_1)}\big). \tag{4.25}$$

Note that

$$\begin{aligned}\big|h_a(t,x)-h_a(t',x')\big| &\le \int_{\mathbb{R}^d}\big|\xi(t,x,y)-\xi(t',x',y)\big|K_a(t,x,y)\,dy \\ &\quad + \int_{\mathbb{R}^d}|\xi(t',x',y)|\big|K_a(t,x,y)-K_a(t',x',y)\big|\,dy =: \mathrm{III}+\mathrm{IV}.\end{aligned} \tag{4.26}$$

<u>Estimate of III:</u> By (4.23) when $|y|\le 1/8$, we have

$$\begin{aligned}&\big|\xi(t,x,y)-\xi(t',x',y)\big| \\ &= \int_0^1\int_0^1 s\big(u(t',x')y^T D^2\eta(t',x'+ss'y)y - u(t,x)y^T D^2\eta(t,x+ss'y)y\big)\,dx\,ds' \\ &\quad + \int_0^1\int_0^1 \Big[y^T D\eta(t',x'+sy)y^T Du(t',x'+s'y) \\ &\qquad\qquad - y^T D\eta(t,x+sy)y^T Du(t,x+s'y)\Big]\,ds\,ds' \\ &\le C|y|^2 l^\alpha\big([u]_{\alpha/\sigma,\alpha;Q_1} + \|u\|_{L_\infty(Q_1)}\big) + C|y|^2 l\big(\|D^2u\|_{L_\infty(Q_{11/8})} + \|Du\|_{L_\infty(Q_{11/8})}\big) \\ &\le C|y|^2 l^\alpha\big(\|u\|_{L_\infty(Q_{11/8})} + \|D^2u\|_{L_\infty(Q_{11/8})}\big),\end{aligned}$$

where we used the interpolation inequalities in the last inequality. On the other hand, when $|y|>1/8$,

$$\big|y^T D\eta(t,x)u(t,x) - y^T D\eta(t',x')u(t',x')\big| \le |y|l^\alpha\|u\|_{\alpha/\sigma,\alpha;Q_1}$$

and

$$\begin{aligned}&\big|(\eta(t,x)-\eta(t,x+y))u(t,x+y) - (\eta(t',x')-\eta(t',x'+y))u(t',x'+y)\big| \\ &= \big|u(t,x+y)\big(\eta(t,x)-\eta(t',x')-\eta(t,x+y)+\eta(t',x'+y)\big) \\ &\quad + \big(\eta(t',x')-\eta(t',x'+y)\big)\big(u(t,x+y)-u(t',x'+y)\big)\big| \\ &\le C(l\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} + l^\alpha\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d}),\end{aligned}$$

which imply that when $|y|>1/8$,

$$\begin{aligned}&\big|\xi(t,x,y)-\xi(t',x',y)\big| \\ &\le C\big(l\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} + l^\alpha\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d}\big) + C|y|l^\alpha\|u\|_{\alpha/\sigma,\alpha;Q_1}.\end{aligned}$$

Now with the above estimates, we obtain

$$\begin{aligned}\mathrm{III} \le &\int_{B_{1/8}} C|y|^2 l^\alpha \big(\|u\|_{L_\infty(Q_{11/8})} + \|D^2 u\|_{L_\infty(Q_{11/8})}\big) K_a(t,x,y)\,dy \\ &+ Cl^\alpha \|u\|_{\alpha/\sigma,\alpha;Q_1} \int_{B^c_{1/8}} |y| K_a(t,x,y)\,dy \\ &+ C\big(l\|u\|_{L_\infty((-1,0)\times\mathbb{R}^d)} + l^\alpha \|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d}\big) \int_{B^c_{1/8}} K_a(t,x,y)\,dy \\ \le\ & Cl^\alpha\big(\|D^2u\|_{L_\infty(Q_{11/8})} + \|u\|_{\alpha/\sigma,\alpha;(-(11/8)^\sigma,0)\times\mathbb{R}^d}\big).\end{aligned}$$

<u>Estimate of IV:</u> By (4.24) and (4.25), we have

$$\mathrm{IV} \le Cl^\alpha A\big(\|u\|_{L_\infty((-(11/8)^\sigma,0)\times\mathbb{R}^d)} + \|Du\|_{L_\infty(Q_{11/8})}\big).$$

The estimates of III and IV with the interpolation inequalities give (4.21). The lemma is proved. ☐

Proof of Theorem 1.1. We apply Theorem 4.1 to v with the estimates of g_a and h_a in Lemma 4.5 to obtain

$$\begin{aligned}[v]_{1+\alpha/\sigma,\alpha+\sigma;Q_{1/2}} \le C\Big(&[v]_{1+\alpha'/\sigma,\alpha'+\sigma} + A[v]_{1+\alpha/\sigma,\alpha+\sigma} + A[v]_{\alpha/\sigma,\alpha} \\ &+ (A+1)(\|u\|_{\alpha/\sigma,\alpha;(-(11/8)^\sigma,0)\times\mathbb{R}^d} + \|D^2u\|_{L_\infty(Q_{11/8})}) + \sup_a [\eta f_a]_{\alpha/\sigma,\alpha;Q_1}\Big).\end{aligned}$$

Since $\eta \equiv 1$ in Q_1 and has compact support in $(-(5/4)^\sigma, (5/4)^\sigma) \times B_{5/4}$, we get

$$\begin{aligned}[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{1/2}} \le C\Big(&[u]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_{5/4}} + C_0 + A[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{5/4}} \\ &+ (A+1)(\|u\|_{\alpha/\sigma,\alpha;(-(11/8)^\sigma,0)\times\mathbb{R}^d} + \|D^2u\|_{L_\infty(Q_{11/8})})\Big).\end{aligned} \tag{4.27}$$

Now we use a scaling argument. For any $\varepsilon > 0$, set $\hat{u}(t,x) := \varepsilon^{-\sigma} u(\varepsilon^\sigma t, \varepsilon x)$. Since u satisfies (1.5), we have

$$\hat{u}_t(t,x) = \inf_a \Big\{ \int_{\mathbb{R}^d} \delta\hat{u}(t,x,y) K^\varepsilon_a(t,x,y)\,dy + f_a(\varepsilon^\sigma t, \varepsilon x)\Big\} \quad \text{in } Q_{1/\varepsilon},$$

where

$$K^\varepsilon_a(t,x,y) = \varepsilon^{d+\sigma} K_a(\varepsilon^\sigma t, \varepsilon x, \varepsilon y).$$

Clearly,

$$\big|K^\varepsilon_a(t,x,y) - K^\varepsilon_a(t',x',y)\big| \le A\varepsilon^\alpha\big(|x-x'|^\alpha + |t-t'|^{\alpha/\sigma}\big)\frac{\Lambda}{|y|^{d+\sigma}}.$$

Then we apply (4.27) to $\hat{u}$ and get

$$\begin{aligned}[\hat{u}]_{1+\alpha/\sigma,\alpha+\sigma;Q_{1/2}} \le C\Big(&[\hat{u}]_{1+\alpha'/\sigma,\alpha+\sigma;Q_{5/4}} + C_0\varepsilon^\alpha + A\varepsilon^\alpha[\hat{u}]_{1+\alpha/\sigma,\alpha+\sigma;Q_{5/4}} \\ &+ (A\varepsilon^\alpha + 1)\big(\|\hat{u}\|_{\alpha/\sigma,\alpha;(-(11/8)^\sigma,0)\times\mathbb{R}^d} + \|D^2\hat{u}\|_{L_\infty(Q_{11/8})}\big)\Big).\end{aligned}$$

Returning back to u, we have

$$\begin{aligned}[u]_{1+\alpha/\sigma,\sigma+\alpha;Q_{\varepsilon/2}} \le C\Big(&\varepsilon^{\alpha'-\alpha}[u]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_{5\varepsilon/4}} + C_0 + A\varepsilon^\alpha[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{5\varepsilon/4}} \\ &+ (A\varepsilon^\alpha+1)\big(\varepsilon^{-\sigma-\alpha}\|u\|_{\alpha/\sigma,\alpha;(-(11\varepsilon/8)^\sigma,0)\times\mathbb{R}^d} + \varepsilon^{2-\sigma-\alpha}\|D^2u\|_{L_\infty(Q_{11\varepsilon/8})}\big)\Big).\end{aligned}$$

By a translation of the coordinates, the inequality above holds for any $(t,x)\in Q_1$ for sufficiently small $\varepsilon>0$

$$\begin{aligned}
&[u]_{1+\alpha/\sigma,\sigma+\alpha;Q_{\varepsilon/2}(t,x)}\\
&\le C\Big(\varepsilon^{\alpha'-\alpha}[u]_{1+\alpha'/\sigma,\alpha'+\sigma;Q_{5\varepsilon/4}(t,x)}+C_0+A\varepsilon^{\alpha}[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{5\varepsilon/4}(t,x)}\\
&\quad+(A\varepsilon^{\alpha}+1)\big(\varepsilon^{-\sigma-\alpha}\|u\|_{\alpha/\sigma,\alpha;(t-(11\varepsilon/8)^{\sigma},t)\times\mathbb{R}^d}+\varepsilon^{2-\sigma-\alpha}\|D^2u\|_{L_\infty(Q_{11\varepsilon/8}(t,x))}\big)\Big).
\end{aligned}\tag{4.28}$$

Let Q^k be defined in (4.1). It is obvious that Q^k monotonically increases to Q_1. Then for any $(t,x),(s,y)\in Q^k$ such that $t\ge s$, we set $l:=\max(|t-s|^{1/\sigma},|x-y|)$. When $l\ge\varepsilon/2$,

$$\begin{aligned}
&\frac{|D^2u(t,x)-D^2u(s,y)|}{l^{\sigma+\alpha-2}}+\frac{|u_t(t,x)-u_t(s,y)|}{l^{\sigma+\alpha-2}}\\
&\le 2^{\sigma+\alpha-1}\varepsilon^{2-\sigma-\alpha}\big(\|u_t\|_{L_\infty(Q^k)}+\|D^2u\|_{L_\infty(Q^k)}\big);
\end{aligned}$$

when $l<\varepsilon/2$,

$$\frac{|D^2u(t,x)-D^2u(s,y)|}{l^{\sigma+\alpha-2}}+\frac{|u_t(t,x)-u_t(s,y)|}{l^{\sigma+\alpha-2}}\le 2[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{\varepsilon/2}(t,x)}.$$

Now we choose $\varepsilon=2^{-k-2}$ so that for any $(t,x)\in Q^k$, $Q_{11\varepsilon/8}(t,x)\subset Q^{k+1}$ and $(t-(11\varepsilon/8)^{\sigma},t)\subset(-1,0)$. Combining the two inequalities above with (4.28), we obtain

$$\begin{aligned}
&[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^k}\le 2^{(k+3)(\sigma+\alpha-2)+1}(\|u_t\|_{L_\infty(Q^k)}+\|D^2u\|_{L_\infty(Q^k)})\\
&\quad+C\Big(2^{(k+2)(\alpha-\alpha')}[u]_{1+\alpha'/\sigma,\alpha'+\sigma;Q^{k+1}}+C_0+2^{-(k+2)\alpha}A[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k+1}}\\
&\quad+(2^{-(k+2)\alpha}A+1)\big(2^{(k+2)(\sigma+\alpha)}\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d}\\
&\quad+2^{(k+2)(\sigma+\alpha-2)}\|D^2u\|_{L_\infty(Q^{k+1})}\big)\Big).
\end{aligned}\tag{4.29}$$

By the interpolation inequalities

$$\begin{aligned}
&[u]_{1+\alpha'/\sigma,\alpha'+\sigma;Q^{k+1}}\le 2^{-2(k+2)(\alpha-\alpha')}[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k+1}}+C2^{2(k+2)(\sigma+\alpha')}\|u\|_{L_\infty},\\
&\|D^2u\|_{L_\infty(Q^{k+1})}+\|u_t\|_{L_\infty(Q^{k+1})}\\
&\le 2^{-2(k+1)(\sigma+\alpha-2)}[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k+1}}+C2^{4(k+1)}\|u\|_{L_\infty(Q^{k+1})},
\end{aligned}$$

we reorganize the right-hand side of (4.29) to get

$$\begin{aligned}
&[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^k}\le C\Big(\big(2^{-(k+1)(\sigma+\alpha-2)}+2^{-(k+2)(\alpha-\alpha')}\big)[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k+1}}\\
&\quad+2^{5k(\sigma+\alpha)}\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d}+C_0\Big),
\end{aligned}$$

where C depends on A. Obviously, $\sigma+\alpha<3$ and there exists a constant k_0 depends on $d,\sigma,\alpha,\lambda,\Lambda$, and A such that $Q_{1/2}\subset Q^{k_0}$ and for any $k\ge k_0$,

$$C\big(2^{-(k+1)(\sigma+\alpha-2)}+2^{-2(k+1)(\alpha-\alpha')}\big)<2^{-16}.$$

Therefore, we have for any $k\ge k_0$,

$$[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^k}\le 2^{-16}[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k+1}}+C2^{15k}\|u\|_{\alpha/\sigma,\alpha}+CC_0.$$

We multiply both sides above by $2^{-16(k-k_0)}$ and then sum from $k=k_0$ to infinity and obtain that

$$[u]_{1+\alpha/\sigma,\alpha+\sigma;Q^{k_0}} \le C2^{15k_0}\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d} + CC_0.$$

In particular,

$$[u]_{1+\alpha/\sigma,\alpha+\sigma;Q_{1/2}} \le C(\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d} + C_0).$$

The proof is completed. $\square$

4.3. **An improved estimate.** By a more careful analysis, we obtain the following corollary when the kernels depend only on y.

Corollary 4.6. *Let $\sigma\in(0,2)$ and $0<\lambda\le\Lambda$. Assume that for any $a\in\mathcal{A}$, K_a only depends on y. There is a constant $\hat{\alpha}\in(0,1)$ depending on d,σ,λ, and Λ so that the following holds. Let $\alpha\in(0,\hat{\alpha})$. Suppose $u\in C^{1+\alpha/\sigma,\sigma+\alpha}(Q_1)\cap C^{\alpha/\sigma,\alpha}((-1,0)\times\mathbb{R}^d)$ is a solution of*

$$u_t = \inf_{a\in\mathcal{A}}(L_a u + f_a) \quad \text{in } Q_1.$$

Then,

$$\begin{aligned} &[u]_{1+\alpha/\sigma,\alpha+\sigma,Q_{1/2}} \\ &\le C\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times B_2} + C\sum_{j=1}^{\infty} 2^{-j\sigma}[u]_{\alpha/\sigma,\alpha;(-1,0)\times(B_{2^{j+1}}\setminus B_{2^j})} + CC_0, \end{aligned} \tag{4.30}$$

where $C_0 = \sup_a [f_a]_{\alpha/\sigma,\alpha;Q_1}$.

Proof. Since the proof is quite similar to the proof of Theorem 1.1, we only provide a sketch here. By a standard scaling and covering argument, we may assume that $u\in C^{1+\alpha/\sigma,\sigma+\alpha}(Q_2)\cap C^{\alpha/\sigma,\alpha}((-2,0)\times\mathbb{R}^d)$ and the equation is satisfied in Q_2. Let η be a cutoff function such that $\eta\in C_0^\infty((-2^\sigma,2^\sigma)\times B_2)$ and $\eta\equiv1$ in $Q_{5/4}$. Let $v=\eta u$, which satisfies

$$v_t = \inf_a(L_a v + h_a + \eta f_a + \eta_t u),$$

where

$$h_a = \int_{\mathbb{R}^d} \xi(t,x,y)K_a(y)\,dy$$

and ξ is defined in (4.22). It is sufficient to estimate $[h_a]_{\alpha/\sigma,\alpha;Q_1}$. Since K_a only depends on y, it follows that

$$|h_a(t,x)-h_a(t',x')| = \mathrm{III},$$

where III is defined in (4.26). The estimate is similar to the one in the proof of Lemma 4.5. For any $(t,x),(t',x')\in Q_1$, since $\eta\equiv1$ in $Q_{5/4}$, $D\eta(t,x)=D\eta(t',x')=0$. When $|y|\le1/4$, $\xi(t,x,y)=0$; When $|y|>1/4$, we have

$$\begin{aligned} &\big|\xi(t,x,y)-\xi(t',x',y)\big| \\ &= \big|(\eta(t,x)-\eta(t,x+y))u(t,x+y) + y^T D\eta(t,x)u(t,x) \\ &\quad - (\eta(t',x')-\eta(t',x'+y))u(t',x'+y) + y^T D\eta(t',x')u(t',x')\big| \\ &\le \big|\eta(t,x+y)u(t,x+y)-\eta(t',x'+y)u(t',x'+y)\big| + \big|u(t,x+y)-u(t',x'+y)\big|. \end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{III} &= \int_{B^c_{1/4}} \big|\xi(t,x,y) - \xi(t',x',y)\big| K_a(y)\,dy \\
&\le Cl^\alpha \|u\|_{\alpha/\sigma,\alpha;(-1,0)\times B_2} + \int_{B^c_{1/4}} \big|u(t,x+y) - u(t',x'+y)\big| K_a(y)\,dy \\
&\le Cl^\alpha \|u\|_{\alpha/\sigma,\alpha;(-1,0)\times B_2} + \sum_{j=-1}^{\infty} \int_{B_{2^j}\setminus B_{2^{j-1}}} |u(t,x+y) - u(t',x'+y)| K_a(y)\,dy \\
&\le Cl^\alpha \|u\|_{\alpha/\sigma,\alpha;(-1,0)\times B_2} + Cl^\alpha \sum_{j=-1}^{\infty} 2^{-j\sigma} [u]_{\alpha/\sigma,\alpha;(-1,0)\times B_{2^{j+1}}} \\
&\le Cl^\alpha \Big(\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times B_2} + \sum_{j=1}^{\infty} 2^{-j\sigma} [u]_{\alpha/\sigma,\alpha;(-1,0)\times (B_{2^j}\setminus B_{2^{j-1}})} \Big),
\end{aligned}$$

which implies that

$$[h_a]_{\alpha/\sigma,\alpha;Q_1} \le C\Big(\|u\|_{\alpha/\sigma,\alpha;(-1,0)\times B_2} + \sum_{j=1}^{\infty} 2^{-j\sigma} [u]_{\alpha/\sigma,\alpha;(-1,0)\times (B_{2^j}\setminus B_{2^{j-1}})} \Big).$$

Then we apply Theorem 1.1 to v and obtain

$$\begin{aligned}
[v]_{1+\alpha/\sigma,\sigma+\alpha;Q_{1/2}} &\le C\Big(\|v\|_{\alpha/\sigma,\alpha;(-1,0)\times\mathbb{R}^d} + \|u\|_{\alpha/\sigma,\alpha;(-1,0)\times B_2} \\
&\quad + \sum_{j=1}^{\infty} 2^{-j\sigma} [u]_{\alpha/\sigma,\alpha;(-1,0)\times (B_{2^j}\setminus B_{2^{j-1}})} + C_0 \Big).
\end{aligned}$$

Combining the fact that $\eta \equiv 1$ in $Q_{5/4}$, we reach (4.30). Therefore, the proof is completed. □

5. Equations with bounded inhomogeneous terms

In this section, we present an application of Corollary 4.6 to nonlocal parabolic equations with merely bounded nonhomogeneous terms:

$$u_t = \inf_{a\in\mathcal{A}} (L_a u + f_a), \tag{5.1}$$

where $\sup_a \|f_a\|_{L_\infty} < \infty$ and

$$L_a u(x) = \int_{\mathbb{R}^d} \delta u(t,x,y) K_a(y)\,dy.$$

Before proving Theorem 1.2, we first show an interpolation inequality involving the Zygmund semi-norm.

Lemma 5.1. *Let $\alpha \in (0,1)$ and $u \in \Lambda^1((-1,0)) \cap L_\infty((-1,0))$. Then we have $u \in C^\alpha((-1,0))$ and*

$$[u]_{\alpha;(-1,0)} \le C\|u\|_{L_\infty((-1,0))} + C[u]_{\Lambda^1((-1,0))}, \tag{5.2}$$

where C depends only on α.

Proof. By mollification, it suffices to prove (5.2) assuming that $u \in C^\alpha((-1,0))$. Let $x, y \in (-1,0)$, $y < x$, and $h := x - y$. When $h > 1/3$,

$$\frac{|u(x) - u(y)|}{h^\alpha} \le 2\cdot 3^\alpha \|u\|_{L_\infty((-1,0))}.$$

When $h < 1/3$, either $x < -1/3$ or $y > -2/3$. If $x < -1/3$, then $2x - y \in (x, 0)$ and

$$\begin{aligned}\frac{|u(x)-u(y)|}{h^\alpha} &\le \frac{1}{2}\frac{|u(2x-y)+u(y)-2u(x)|}{h^\alpha} + \frac{1}{2}\frac{|u(2x-y)-u(y)|}{h^\alpha} \\ &\le \frac{3^{\alpha-1}}{2}[u]_{\Lambda^1((-1,0))} + \frac{1}{2^{1-\alpha}}[u]_{\alpha;(-1,0)}.\end{aligned}$$

The case when $y > -2/3$ is similar. Therefore,

$$[u]_{\alpha;(-1,0)} \le 2\cdot 3^\alpha \|u\|_{L_\infty((-1,0))} + \frac{1}{2^{1-\alpha}}[u]_{\alpha;(-1,0)} + \frac{3^{\alpha-1}}{2}[u]_{\Lambda^1((-1,0))},$$

which yields (5.2). □

In the lemma below, we prove that the Zygmund norm of the odd extension of a function is bounded by its original Zygmund norm. It is well known that the same result holds if we replace the Zygmund norm by any Hölder norm.

Lemma 5.2. *Assume that $u \in \Lambda^1((-\infty, 0])$ and $u(0) = 0$. Let $\tilde{u}$ be the odd extension of u. Then*

$$[\tilde{u}]_{\Lambda^1(\mathbb{R})} \le 3[u]_{\Lambda^1((-\infty,0))}.$$

Proof. By the definition, we need to estimate

$$h^{-1}\big|\tilde{u}(x+h) + \tilde{u}(x-h) - 2\tilde{u}(x)\big|, \tag{5.3}$$

where $h > 0$. Clearly, when $x+h, x, x-h \in (-\infty, 0)$, or $x+h, x-h, x \in (0,\infty)$, (5.3) is bounded by $[u]_{\Lambda^1(-\infty,0)}$. Since $\tilde{u}$ is the odd extension of u, when $x = 0$ $\tilde{u}(h) + \tilde{u}(-h) = 0$. It remains to consider the case that these $x + h > 0$ and $x, x-h < 0$, or $x+h, x > 0$ and $x - h < 0$. Without loss of generality, we assume that $x + h > 0$ and $x, x-h < 0$, which implies $x \in (-h, 0)$. Then

$$\begin{aligned}&\tilde{u}(x+h) + \tilde{u}(x-h) - 2\tilde{u}(x) = -u(-x-h) + u(x-h) - 2u(x) \\ &= (u(-x-h) + u(x-h) - 2u(-h)) - 2(u(-x-h) + u(x) - 2u(-h/2)) \\ &\quad + 2(u(-h) + u(0) - 2u(-h/2)) \\ &\le [u]_{\Lambda^1(-\infty,0)}(|x| + 2|x+h/2| + 2|h/2|) \le 3[u]_{\Lambda^1(-\infty,0)}h.\end{aligned}$$

Therefore, the lemma is proved. □

Let η be a smooth even function in $\mathbb{R}$ with unit integral and vanishing outside $(-1, 1)$. For $R > 0$, we define the mollification of u with respect to t as

$$u^{(R)}(t,x) = \int_{\mathbb{R}} u(t+s,x)R^{-\sigma}\eta(sR^{-\sigma})\,ds.$$

The following lemmas will also be used in our proof.

Lemma 5.3. *Let $\sigma \in (1,2)$, $\alpha \in (0,1)$, and $R > 0$ be constants. Assume that u defined on $\mathbb{R}^{d+1}$ is C^σ in x, Λ^1 in t, and Du is $C^{(\sigma-1)/\sigma}$ in t. Let $p = p(t,x)$ be the first-order Taylor expansion of $u^{(R)}$ at the origin. Then for any integer $j \ge 0$, we have*

$$\begin{aligned}&[u-p]^*_{\alpha;(-R^\sigma,0)\times B_{2^jR}} \\ &\le C(2^jR)^{\sigma-\alpha}[u]^*_\sigma + C2^{(1-\alpha)j}R^{\sigma-\alpha}[Du]^t_{\frac{\sigma-1}{\sigma}} + C2^{-j\alpha}R^{\sigma-\alpha}[u]^t_{\Lambda^1}\end{aligned} \tag{5.4}$$

and

$$[u-p]^t_{\alpha/\sigma;(-R^\sigma,0)\times B_{2^jR}} \le C2^{j(\sigma-\alpha/2)}R^{\sigma-\alpha}[u]^*_\sigma + C2^{j(\sigma+1-\alpha)/2}R^{\sigma-\alpha}[Du]^t_{\frac{\sigma-1}{\sigma}} + C2^{j(\sigma-\alpha/2)}R^{\sigma-\alpha}[u]^t_{\Lambda^1}, \tag{5.5}$$

where $C>0$ is a constant depending only on d, σ, and α.

Proof. We first estimate the Hölder semi-norm in x. By the interpolation inequality,

$$[u-p]^*_{\alpha;(-R^\sigma,0)\times B_{2^jR}} \le (2^jR)^{-\alpha}\|u-p\|_{L_\infty((-R^\sigma,0)\times B_{2^jR})} + (2^jR)^{\sigma-\alpha}[u-p]^*_{\sigma;(-R^\sigma,0)\times B_{2^jR}}. \tag{5.6}$$

Because p is linear,

$$[u-p]^*_{\sigma;(-R^\sigma,0)\times B_{2^jR}} = [u]^*_{\sigma;(-R^\sigma,0)\times B_{2^jR}}. \tag{5.7}$$

Since η is even with unit integral, by Lemma 5.2 we have

$$\big|u(t,x)-u^{(R)}(t,x)\big| = \Big|\int_{\mathbb{R}}\Big(\frac{u(t+s,x)+u(t-s,x)}{2}-u(t,x)\Big)R^{-\sigma}\eta(sR^{-\sigma})\,ds\Big| \le CR^\sigma[u]^t_{\Lambda^1}. \tag{5.8}$$

Furthermore, for any $(t,x)\in(-R^\sigma,0)\times B_{2^jR}$,

$$\begin{aligned}
&\big|u^{(R)}(t,x)-p(t,x)\big| = \big|u^{(R)}(t,x)-u^{(R)}(0,0)-\partial_tu^{(R)}(0,0)t-x^TDu^{(R)}(0,0)\big|\\
&\le \big|u^{(R)}(t,x)-u^{(R)}(t,0)-x^TDu^{(R)}(t,0)\big|\\
&\quad+\big|u^{(R)}(t,0)-u^{(R)}(0,0)-\partial_tu^{(R)}(0,0)t\big| + \big|x^TDu^{(R)}(0,0)-x^TDu^{(R)}(t,0)\big|\\
&\le (2^jR)^\sigma[u]^*_\sigma + R^{2\sigma}\big\|\partial_t^2u^{(R)}\big\|_{L_\infty(-R^\sigma,0)\times B_{2^jR}} + C2^jR^\sigma\big[Du^{(R)}\big]^t_{\frac{\sigma-1}{\sigma}}.
\end{aligned}$$

Integrating by part and noting that η'' is an even function and $\int\eta''=0$, we obtain

$$\begin{aligned}
\big|\partial_t^2u^{(R)}\big| &= \Big|\int_{\mathbb{R}}\partial_t^2u(t+s,x)R^{-\sigma}\eta(sR^{-\sigma})\,ds\Big|\\
&= \Big|\int_{\mathbb{R}^d}\Big(\frac{u(t+s,x)+u(t-s,x)}{2}-u(t,x)\Big)R^{-3\sigma}\eta''(sR^{-\sigma})\,ds\Big| \le CR^{-\sigma}[u]^t_{\Lambda^1}.
\end{aligned}$$

Therefore, combining the two inequalities above, we have

$$\big\|u^{(R)}-p\big\|_{L_\infty((-R^\sigma,0)\times B_{2^jR})} \le C(2^jR)^\sigma[u]^*_\sigma + C2^jR^\sigma[Du]^t_{\frac{\sigma-1}{\sigma}} + CR^\sigma[u]^t_{\Lambda^1},$$

which together with (5.8) implies that

$$\|u-p\|_{L_\infty((-R^\sigma,0)\times B_{2^jR})} \le C(2^jR)^\sigma[u]^*_\sigma + C2^jR^\sigma[Du]^t_{\frac{\sigma-1}{\sigma}} + CR^\sigma[u]^t_{\Lambda^1}. \tag{5.9}$$

We plug (5.7) and (5.9) in (5.6) and get (5.4).

Next we estimate the Hölder semi-norm in t. Obviously,

$$[u-p]^t_{\alpha/\sigma;(-R^\sigma,0)\times B_{2^jR}} \le [u-p]^t_{\alpha/\sigma;(-2^{j\sigma/2}R^\sigma,0)\times B_{2^jR}}.$$

From Lemma 5.1 and scaling, we have

$$\begin{aligned}
&[u-p]^t_{\alpha/\sigma;(-2^{j\sigma/2}R^\sigma,0)\times B_{2^jR}}\\
&\le C(2^{j/2}R)^{-\alpha}\|u-p\|_{L_\infty((-2^{j\sigma/2}R^\sigma,0)\times B_{2^jR})} + C(2^{j/2}R)^{\sigma-\alpha}[u-p]^t_{\Lambda^1}\\
&\le C(2^{j/2}R)^{-\alpha}\|u-p\|_{L_\infty((-2^{j\sigma/2}R^\sigma,0)\times B_{2^jR})} + C(2^{j/2}R)^{\sigma-\alpha}[u]^t_{\Lambda^1}.
\end{aligned}$$

We follow (5.9) to estimate

$$\begin{aligned}
&\|u-p\|_{L_\infty((-2^{j\sigma/2}R^\sigma,0)\times B_{2^jR})}\\
&\le \|u-u^{(R)}\|_{L_\infty((-2^{j\sigma/2}R^\sigma,0)\times B_{2^jR})}+\|u^{(R)}-p\|_{L_\infty((-2^{j\sigma/2}R^\sigma,0)\times B_{2^jR})}\\
&\le CR^\sigma[u]^t_{\Lambda^1}+C(2^jR)^\sigma[u]^*_\sigma+C2^{j\sigma}R^\sigma[u]^t_{\Lambda^1}+C2^{j(\sigma+1)/2}R^\sigma[Du]^t_{\frac{\sigma-1}{\sigma}}.
\end{aligned}$$

Therefore, we reach (5.5). The lemma is proved. □

In the sequel, we set

$$[u]^t_{\Lambda^1}:=[u]^t_{\Lambda^1(\mathbb{R}^{d+1}_0)},\quad [u]^*_\sigma:=[u]^*_{\sigma;\mathbb{R}^{d+1}_0},\quad\text{and}\quad [Du]^t_{\frac{\sigma-1}{\sigma}}=:[Du]^t_{\frac{\sigma-1}{\sigma};\mathbb{R}^{d+1}_0}.$$

Define $\mathcal{P}_0$ to be the set of first-order polynomials of t, and $\mathcal{P}_1$ to be the set of first-order polynomials of t,x.

Lemma 5.4. *(i) When $\sigma\in(0,1)$, we have*

$$[u]^t_{\Lambda^1}+[u]^*_\sigma\le C\sup_{r>0}\sup_{(t,x)\in\mathbb{R}^{d+1}_0}r^{-\sigma}\inf_{p\in\mathcal{P}_0}\|u-p\|_{L_\infty(Q_r(t,x))},$$

where $C>0$ is a constant depending only on d and σ.

(ii) When $\sigma\in(1,2)$, we have

$$[u]^t_{\Lambda^1}+[u]^*_\sigma+[Du]^t_{\frac{\sigma-1}{\sigma}}\le C\sup_{r>0}\sup_{(t,x)\in\mathbb{R}^{d+1}_0}r^{-\sigma}\inf_{p\in\mathcal{P}_1}\|u-p\|_{L_\infty(Q_r(t,x))},$$

where $C>0$ is a constant depending only on d and σ.

Proof. The estimates of $[u]^*_\sigma$ and $[Du]^t_{\frac{\sigma-1}{\sigma}}$ are standard. See, for instance, [18, Section 3.3]. We only consider $[u]^t_{\Lambda^1}$. For any polynomial p which is linear in t, by the triangle inequality,

$$\begin{aligned}
&\big|u(t+s,x)+u(t-s,x)-2u(t,x)\big|\\
&=\big|u(t+s,x)-p(t+s,x)+u(t-s,x)-p(t-s,x)-2(u(t,x)-p(t,x))\big|\\
&\le 4\|u-p\|_{L_\infty(Q_r(t+s,x))},
\end{aligned}$$

where $r^\sigma=2s$. Since p is arbitrary, the inequality above implies that

$$\big|u(t+s,x)+u(t-s,x)-2u(t,x)\big|\le 8sr^{-\sigma}\inf_p\|u-p\|_{L_\infty(Q_r(t+s,x))}.$$

The lemma is proved. □

Proof of Theorem 1.2. We only treat the case when $\sigma>1$. For the case when $\sigma<1$, the proof is almost the same with minor modifications.

We extend u to $\{t>0\}$ by defining

$$u(t,x)=2u(0,x)-u(-t,x)\quad\text{for } t>0$$

By Lemma 5.2,

$$[u]^t_{\Lambda^1(\mathbb{R}^{d+1})}\le C[u]^t_{\Lambda^1}.$$

Let $\hat\alpha$ be the constant in Corollary 4.6 and $\alpha\in(0,\hat\alpha)$. Let $R>0$ be a constant and p be defined as in Lemma 5.3. Let $K\ge 2\|u-p\|_{L_\infty(Q_{2R})}$ be a constant to be specified later and denote $g_K=\max\big(\min(u-p,K),-K\big)$. Clearly, $g_K\in C^{\alpha/\sigma,\alpha}$ and any $C^{\alpha/\sigma,\alpha}$ norm (or semi-norm) of g_K is less than or equal to that of $u-p$.

Let v_K be the solution to

$$\begin{cases}\partial_t v_K = \inf_a (L_a v_K) & \text{in } Q_{2R},\\ v_K = g_K & \text{in } \mathbb{R}^{d+1}_0 \setminus Q_{2R}.\end{cases} \tag{5.10}$$

The solvability follows from Theorem 1.1 and a regularization argument; see [2, 23]. We apply Corollary 4.6 to v_K with a scaling to get

$$\begin{aligned}&[v_K]_{1+\alpha/\sigma,\alpha+\sigma;Q_{R/2}}\\ &\le C\Big(R^{-\alpha-\sigma}\|v_K\|_{L_\infty((-R^\sigma,0)\times B_{2R})} + R^{-\sigma}[v_K]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times B_{2R}}\\ &\quad + \sum_{j=2}^\infty 2^{-j\sigma}R^{-\sigma}[v_K]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times(B_{2^jR}\setminus B_{2^{j-1}R})}\Big)\\ &\le C\Big(R^{-\alpha-\sigma}\|v_K\|_{L_\infty((-R^\sigma,0)\times B_{2R})} + R^{-\sigma}[v_K]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times B_{2R}}\\ &\quad + \sum_{j=2}^\infty 2^{-j\sigma}R^{-\sigma}[u-p]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times(B_{2^jR}\setminus B_{2^{j-1}R})}\Big).\end{aligned} \tag{5.11}$$

where in the last equality we used the fact that $v_K = g_K$ in $(-(2R)^\sigma, 0)\times B^c_{2R}$ and the Hölder norm of g_K is less than the Hölder norm of $u-p$.

By Lemma 5.3,

$$\begin{aligned}[u-p]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times B_{2^jR}} &\le C2^{j(\sigma-\alpha/2)}R^{\sigma-\alpha}[u]^t_{\Lambda^1}\\ &\quad + C2^{j(\sigma-\alpha/2)}R^{\sigma-\alpha}[u]^*_\sigma + C2^{j(\sigma+1-\alpha)/2}R^{\sigma-\alpha}[Du]^t_{\frac{\sigma-1}{\sigma}},\end{aligned} \tag{5.12}$$

which together with (5.11) gives

$$\begin{aligned}&[v_K]_{1+\alpha/\sigma,\alpha+\sigma;Q_{R/2}}\\ &\le C\big(R^{-\alpha-\sigma}\|v_K\|_{L_\infty((-R^\sigma,0)\times B_{2R})} + R^{-\sigma}[v_K]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times B_{2R}}\\ &\quad + R^{-\alpha}[u]^t_{\Lambda^1} + R^{-\alpha}[u]^*_\sigma + R^{-\alpha}[Du]^t_{\frac{\sigma-1}{\sigma}}\big).\end{aligned} \tag{5.13}$$

Next we estimate $w_K := g_K - v_K$, which is equal to $u-p-v_K$ in Q_{2R} by the choice of K. By (5.1) and (5.10), w_K satisfies

$$\begin{cases}\partial_t w_K \le \mathcal{M}^+ w_K + h_K + \sup_{a\in\mathcal{A}}\|f_a\|_{L_\infty} & \text{in } Q_{2R},\\ \partial_t w_K \ge \mathcal{M}^- w_K + \hat h_K - \sup_{a\in\mathcal{A}}\|f_a\|_{L_\infty} & \text{in } Q_{2R},\\ w_K = 0 & \text{in } \mathbb{R}^{d+1}_0\setminus Q_{2R},\end{cases}$$

where

$$h_K := \mathcal{M}^+(u-p-g_K), \quad \hat h_K := \mathcal{M}^-(u-p-g_K).$$

By the dominated convergence theorem, it is not hard to see that

$$\|h_K\|_{L_\infty(Q_{2R})},\ \|\hat h_K\|_{L_\infty(Q_{2R})} \to 0 \quad \text{as} \quad K\to\infty.$$

We then fix K large enough so that

$$\|h_K\|_{L_\infty(Q_{2R})} + \|\hat h_K\|_{L_\infty(Q_{2R})} \le \sup_{a\in\mathcal{A}}\|f_a\|_{L_\infty}.$$

From Lemma 2.5, we have

$$\|w_K\|_{L_\infty(Q_{2R})} \le CR^\sigma \sup_{a\in\mathcal{A}}\|f_a\|_{L_\infty}, \quad [w_K]_{\alpha/\sigma,\alpha;Q_{2R}} \le CR^{\sigma-\alpha}\sup_{a\in\mathcal{A}}\|f_a\|_{L_\infty}, \tag{5.14}$$

where C depends on d,σ,λ, and Λ.

Now let q_K be the first-order Taylor expansion of v_K at the origin. Then by (5.13), for any $r \in (0, R/2)$,

$$\begin{aligned}
&\|u - p - q_K\|_{L_\infty(Q_r)} \le \|u - p - v_K\|_{L_\infty(Q_r)} + \|v_K - q_K\|_{L_\infty(Q_r)} \\
&\le \|u - p - v_K\|_{L_\infty(Q_r)} + C(r/R)^{\sigma+\alpha}\|v_K\|_{L_\infty((-R^\sigma,0)\times B_{2R}))} \\
&+ Cr^{\sigma+\alpha}R^{-\sigma}[v_K]_{\alpha/\sigma,\alpha;(-R^\sigma,0)\times B_{2R})} + Cr^{\sigma+\alpha}R^{-\alpha}\big([u]^t_{\Lambda^1} + [u]^*_\sigma + [Du]^t_{\frac{\sigma-1}{\sigma}}\big).
\end{aligned} \tag{5.15}$$

Since $v_K = u - p + w_K$ in Q_{2R}, we plug (5.9) and (5.12) with $j = 0$, and (5.14) to (5.15) and obtain

$$\|u - p - q_K\|_{L_\infty(Q_r)} \le CR^\sigma \sup_{a\in\mathcal{A}} \|f_a\|_{L_\infty} + Cr^{\sigma+\alpha}R^{-\alpha}\big([u]^t_{\Lambda^1} + [u]^*_\sigma + [Du]^t_{\frac{\sigma-1}{\sigma}}\big).$$

Dividing both sides of the inequality above by r^σ, we have

$$\begin{aligned}
&r^{-\sigma}\|u - p - q_K\|_{L_\infty(Q_r)} \\
&\le C(R/r)^\sigma \sup_{a\in\mathcal{A}} \|f_a\|_{L_\infty} + C(r/R)^\alpha\big([u]^t_{\Lambda^1} + [u]^*_\sigma + [Du]^t_{\frac{\sigma-1}{\sigma}}\big).
\end{aligned}$$

Set $r = R/M$, where $M \ge 2$ is a constant to be determined. Note that the center of the cylinder can be replaced by any point (t, x) in $\mathbb{R}^{d+1}_0$, i.e.,

$$\begin{aligned}
&r^{-\sigma}\|u - p - q_K\|_{L_\infty(Q_r(t,x))} \\
&\le CM^\sigma \sup_{a\in\mathcal{A}} \|f_a\|_{L_\infty} + CM^{-\alpha}\big([u]^t_{\Lambda^1} + [u]^*_\sigma + [Du]^t_{\frac{\sigma-1}{\sigma}}\big),
\end{aligned}$$

which together with Lemma 5.4 implies

$$\begin{aligned}
&[u]^t_{\Lambda^1} + [u]^*_\sigma + [Du]^t_{\frac{\sigma-1}{\sigma}} \\
&\le C \sup_{r>0} \sup_{(t,x)\in\mathbb{R}^{d+1}_0} r^{-\sigma} \inf_{p\in\mathcal{P}_1} \|u - p\|_{L_\infty(Q_r(t,x))} \\
&\le CM^\sigma \sup_{a\in\mathcal{A}} \|f_a\|_{L_\infty} + CM^{-\alpha}\big([u]^t_{\Lambda^1} + [u]^*_\sigma + [Du]^t_{\frac{\sigma-1}{\sigma}}\big).
\end{aligned} \tag{5.16}$$

By taking M sufficiently large in (5.16) so that $CM^{-\alpha} < 1/2$, we obtain

$$[u]^t_{\Lambda^1} + [u]^*_\sigma + [Du]^t_{\frac{\sigma-1}{\sigma}} \le CM^\sigma \sup_{a\in\mathcal{A}} \|f_a\|_{L_\infty}.$$

The theorem is proved. □

The proof of Corollary of 1.3 is similar to that of Theorem 1.1, and thus omitted.

APPENDIX

In the appendix, we provide a sketch of the proof of Corollary 2.2.

Proof. By a scaling argument, we assume that $r = 1$. Let $k \ge 1$ be a constant to be determined later. Set $\hat{\delta} = \delta/k$. Let $(t_0, x_0) \in \overline{Q_{\delta/2}}$ be such that $u(t_0, x_0) = \inf_{Q_{\delta/2}} u$. Since $\sigma \in (1, \infty)$, we have $2^{-\sigma} \le 1 - 4^{-\sigma}$. By a scaling and translation of the coordinates, we apply Proposition 2.1 to u in $Q_{\hat{\delta}}(t_0, x_0)$ and obtain

$$\hat{\delta}^{-(\sigma+d)/\varepsilon}\|u\|_{L_\varepsilon(\hat{Q}_1)} \le C_1\Big(\inf_{Q_{\hat{\delta}}(t_0,x_0)} u + C\hat{\delta}^\sigma\Big),$$

where $\hat{Q}_1 = Q_{\hat\delta}(t_1, x_0)$ and $t_1 = t_0 - (4^\sigma - 1)\hat\delta^\sigma$. For any $x_1 \in B_{\hat\delta/2}(x_0)$,

$$\begin{aligned}\|u\|_{L_\varepsilon(\hat{Q}_1)} &\ge \|u\|_{L_\varepsilon((t_1-\delta^\sigma,t_1)\times B_{\hat\delta/2}(x_1))}\\ &\ge C_2\hat\delta^{(\sigma+d)/\varepsilon}\inf_{(t_1-\delta^\sigma,t_1)\times B_{\hat\delta/2}(x_1)} u \ge C_2\hat\delta^{(\sigma+d)/\varepsilon}\inf_{Q_{\hat\delta}(t_1,x_1)} u,\end{aligned}$$

where $C_2 > 0$ depending only on d. Therefore,

$$\inf_{Q_{\hat\delta}(t_1,x_1)} u \le C_1/C_2\Big(\inf_{Q_{\hat\delta}(t_0,x_0)} u + C\hat\delta^\sigma\Big).$$

Applying Proposition 2.1 again, we have

$$\hat\delta^{-(\sigma+d)/\varepsilon}\|u\|_{L_\varepsilon(\hat{Q}_2)} \le C_1\Big(\inf_{Q_{\hat\delta}(t_1,x_1)} u + C\hat\delta^\sigma\Big),$$

where $\hat{Q}_2 = Q_{\hat\delta}(t_2, x_1)$ and $t_2 = t_0 - 2(4^\sigma - 1)\hat\delta^\sigma$, and for any $x_2 \in B_{\hat\delta/2}(x_1)$,

$$\inf_{Q_{\hat\delta}(t_2,x_2)} u \le C_1/C_2\Big(\inf_{Q_{\hat\delta}(t_1,x_1)} u + C\hat\delta^\sigma\Big).$$

By induction, or any $x_{n-1} \in B_{(n-1)\hat\delta/2}(x_0) \cap B_1$,

$$\begin{aligned}\hat\delta^{-(\sigma+d)/\varepsilon}\|u\|_{L_\varepsilon(\hat{Q}_n)} &\le C_3\Big(\inf_{Q_{\hat\delta}(t_0,x_0)} u + C\hat\delta^\sigma\Big)\\ &\le C_3\big(u(t_0,x_0) + C\hat\delta^\sigma\big) = C_3\Big(\inf_{Q_{\delta/2}} u + C\hat\delta^\sigma\Big),\end{aligned}$$

where $\hat{Q}_n = Q_{\hat\delta}(t_n, x_{n-1})$, $t_n = t_0 - n(4^\sigma - 1)\hat\delta^\sigma$, and C_3 is a constant depending only on λ, Λ, d, and n. Notice that $|x_0| \le \delta/2$, $t_0 \in [-(\delta/2)^\sigma, 0]$, and $\sigma > 1$. We can choose $k \ge 1$ in a suitable range depending only on σ and δ, and then $n \le [2k/\delta] + 1$, such that $\hat{Q}_n$ runs through $(-\delta^\sigma, -\delta^\sigma + (4^\sigma - 1)\hat\delta^\sigma) \times B_1$. Finally, by applying Proposition 2.1 again and using a simple covering argument, we prove the corollary. □

References

[1] Luis Caffarelli and Luis Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.*, 62(5):597–638, 2009.

[2] Luis Caffarelli and Luis Silvestre. The Evans-Krylov theorem for nonlocal fully nonlinear equations. *Ann. of Math. (2)*, 174(2):1163–1187, 2011.

[3] Luis Caffarelli and Luis Silvestre. Regularity results for nonlocal equations by approximation. *Arch. Ration. Mech. Anal.*, 200(1):59–88, 2011.

[4] Luis A. Caffarelli and Xavier Cabré. *Fully nonlinear elliptic equations*, volume 43 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1995.

[5] H. Chang-Lara and G. Davila. $C^{\sigma+\alpha}$ estimates for concave, non-local parabolic equations with critical drift. *ArXiv e-prints*, August 2014.

[6] H. Chang-Lara and G. Davila. Hölder estimates for non-local parabolic equations with critical drift. *ArXiv e-prints*, August 2014.

[7] Héctor Chang-Lara and Gonzalo Dávila. Regularity for solutions of non local parabolic equations. *Calc. Var. Partial Differential Equations*, 49(1-2):139–172, 2014.

[8] Héctor Chang-Lara and Gonzalo Dávila. Regularity for solutions of nonlocal parabolic equations II. *J. Differential Equations*, 256(1):130–156, 2014.

[9] Rama Cont and Peter Tankov. *Financial modelling with jump processes*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[10] Hongjie Dong and Doyoon Kim. On L_p-estimates for a class of non-local elliptic equations. *J. Funct. Anal.*, 262(3):1166–1199, 2012.

[11] Hongjie Dong and Doyoon Kim. Schauder estimates for a class of non-local elliptic equations. *Discrete Contin. Dyn. Syst.*, 33(6):2319–2347, 2013.

[12] Lawrence C. Evans. Classical solutions of fully nonlinear, convex, second-order elliptic equations. *Comm. Pure Appl. Math.*, 35(3):333–363, 1982.

[13] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[14] Tianling Jin and Jingang Xiong. Schauder estimates for nonlocal fully nonlinear equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, pages –, 2015.

[15] Tianling Jin and Jingang Xiong. Schauder estimates for solutions of linear parabolic integro-differential equations. *Discrete Contin. Dyn. Syst.*, 35(12):5977–5998, 2015.

[16] Takashi Komatsu. Pseudodifferential operators and Markov processes. *J. Math. Soc. Japan*, 36(3):387–418, 1984.

[17] N. V. Krylov. Boundedly inhomogeneous elliptic and parabolic equations in a domain. *Izv. Akad. Nauk SSSR Ser. Mat.*, 47(1):75–108, 1983.

[18] N. V. Krylov. *Lectures on elliptic and parabolic equations in Hölder spaces*, volume 12 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1996.

[19] R. Mikulevičius and H. Pragarauskas. On the Cauchy problem for certain integro-differential operators in Sobolev and Hölder spaces. *Liet. Mat. Rink.*, 32(2):299–331, 1992.

[20] R. Mikulevicius and H. Pragarauskas. On the Cauchy problem for integro-differential operators in Hölder classes and the uniqueness of the martingale problem. *Potential Anal.*, 40(4):539–563, 2014.

[21] M. V. Safonov. Classical solution of second-order nonlinear elliptic equations. *Izv. Akad. Nauk SSSR Ser. Mat.*, 52(6):1272–1287, 1328, 1988.

[22] R. W. Schwab and L. Silvestre. Regularity for parabolic integro-differential equations with very irregular kernels. *ArXiv e-prints*, December 2014.

[23] Joaquim Serra. $C^{\sigma+\alpha}$ regularity for concave nonlocal fully nonlinear elliptic equations with rough kernels. *Calc. Var. Partial Differential Equations*, 54(4):3571–3601, 2015.

(H. Dong) Division of Applied Mathematics, Brown University, 182 George Street, Providence, RI 02912, USA

E-mail address: `Hongjie_Dong@brown.edu`

(H. Zhang) Division of Applied Mathematics, Brown University, 182 George Street, Providence, RI 02912, USA

E-mail address: `Hong_Zhang@brown.edu`